Some Noetherian Rings

Theorem 1 *If A is a commutative Noetherian ring, then so is $A[X]$.*

Proof Let $I \subset A[X]$ be an ideal. For each $i \geq 0$, let $I_i \subset A$ denote the ideal generated by the coefficients of X^i of all degree-i polynomials in the ideal I. Since I is closed under multiplication by X, we have $I_0 \subset I_1 \subset I_2 \subset \cdots$. Since A is Noetherian, this chain stabilizes, that is, for some r, $I_{r+i} = I_r$ for all $i > 0$. For each $i \leq r$, choose a finite set of generators for the ideal I_i, say $I_i = (a_{ij})$, and choose polynomials $f_{ij}(X) \in I$ of degree i with leading coefficient a_{ij}.

I claim that $I = (f_{ij}(X)) \subset A[X]$. Namely, suppose $f(X) \in I$ has degree d. If $d = 0$, $f(X)$ is just an element of A belonging to the ideal (a_{0j}). These constants are included in our proposed set of generators for I. If $0 < d \leq r$ and if a_d is the leading coefficient of $f(X)$, write $a_d = \sum_j c_{dj} a_{dj} \in I_d$. Then $f(X) - \sum_j c_{dj} f_{dj}(X) \in I$. But this polynomial has degree $< d$, since the degree-d coefficients cancel out. By induction $f(X) \in (f_{ij}(X))$. Finally, suppose $d > r$. Since $I_d = I_r$, write $a_d = \sum_j c_{rj} a_{rj}$. Then $f(X) - \sum_j c_{rj} X^{d-r} f_{rj}(X) \in I$. Again, this polynomial has degree less than d, so we are finished by induction. ∎

Theorem 2 *If A is a commutative Noetherian ring, then so is $A[[X]]$.*

Proof For a power series, define the **degree** to be the *least* power of X which occurs in the series. Call the coefficient of that least power of X the **leading coefficient**. Again, if $I \subset A[[X]]$ is an ideal, let $I_i \subset A$ denote the ideal generated by leading coefficients of power series of degree i which belog to I. Multiplication by X shows that $I_0 \subset I_1 \subset I_2 \subset \cdots$. Again, for some r, $I_{r+i} = I_r$ for all $i > 0$. For $i \leq r$, choose a finite set of generators $I_i = (a_{ij})$ and choose power series $f_{ij}(X) \in I$ of degree i with leading coefficient a_{ij}.

I claim that $I = (f_{ij}(X)) \subset A[[X]]$. Namely, if $f(X) \in I$ has degree $d < r$, there will be a finite sum $g(X) = f(X) - \sum_{ij} c_{ij} f_{ij}(X) \in I$, $d \leq i < r$, which has degree $\geq r$. But now, since $I_{r+i} = I_r$ for all $i > 0$, it is clear that we can write $g(X) = \sum_j (\sum_{i \geq 0} d_{ij} X^i) f_{rj}(X) \in A[[X]]$. Namely, just choose the coefficients d_{ij} inductively for $i \geq 0$ so as to force the right-hand side to agree with $g(X)$ through degree $r + i$. The two formulas for $g(X)$ show $f(X) \in (f_{ij}(X)) \subset A[[X]]$. ∎