The Wussy Weak Nullstellensatz

The weak Nullstellensatz says that if F is a subfield of an algebraically closed field K, then any set of polynomials that generate a proper ideal, I, in $F[X_1, \ldots, X_n]$ have a common zero in K^n. Here I will prove this under the additional assumption that K contains an infinite set of elements that are algebraically independent over the prime subfield, $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$. (Such a K was classically called a 'universal domain'.)

Since the complex numbers, $\mathbb{C}$, are uncountable, clearly $\mathbb{C}$ contains an uncountably infinite set of elements algebraically independent over $\mathbb{Q}$. Therefore, this wussy Nullstellensatz implies the weak Nullstellensatz for $K = \mathbb{C}$.

A key step of the proof is an extremely simple observation.

TRIVIAL OBSERVATION: If k is a subfield of K and if $k[X_1, \ldots, X_n]$ is the polynomial ring and if P is an ideal in $k[X_1, \ldots, X_n]$, then a homomorphism $h : k[X_1, \ldots, X_n]/P \to K$ that agrees with the inclusion of k into K on constants, is *exactly the same thing* as a common zero in K^n of all polynomials in ideal P.

Why? In general a ring homomorphism $h : A/P \to B$ is the same thing as a homomorphism $h : A \to B$ with P contained in the kernel. If A is a polynomial ring, a homomorphism $h : k[X_1, \ldots, X_n] \to K$ is uniquely determined by where the constants in k go and by the images $h(X_j) = b_j$ in K. We've agreed above in our case that the constants in k are just included in K by an identity map. If you know where the X_j go by h, you know where every polynomial in the X_j goes by h, namely $h(f(X_1, \ldots, X_n)) = f(b_1, \ldots, b_n)$. In other words, *evaluate* f at the vector $\mathbf{b} = (b_j)$. This is utterly trivial. If X, Y, Z go to a, b, c, then where does, say, $f(X, Y, Z) = X^2YZ^3 + XY^2 + Z$ go? You tell me, but you better tell me it goes to $a^2bc^3 + ab^2 + c = f(a, b, c)$, and, not only that, it is an utter triviality that it goes to $f(a, b, c)$. So, if ideal P is in the kernel of h, then the vector $\mathbf{b}$ is a common zero of every polynomial in P.

How is this trivial observation then used to prove the weak Nullstellensatz under the additional assumption that K has infinite transcendence degree over the prime field? Going back to the notation in the first paragraph, let k denote the smallest subfield of K containing the coefficients of some finite set of generators of ideal I. It is true that a necessary step is the "lemma" that K still has infinite transcendence degree over k, but this is rather easy. (And really trivial, by uncountability, if $K = \mathbb{C}$.) Now choose any prime ideal P in $k[X_1, \ldots, X_n]$ containing these generators of I. Then $k[X_1, \ldots, X_n]/P$ is an integral domain. Call its field of fractions L. By the trivial observation above, we will have our desired common zero of the generators of I, (in fact, a common zero of all polynomials in P) if we can find an embedding of L into K, which extends the given inclusion of k into K.

Well, WLOG we can assume $\{X_1, \ldots, X_r\}$ are algebraically independent over k and that L is an algebraic extension of its subfield $k(x_1, \ldots, x_r)$, where $x_j = X_j \bmod P$. This subfield $k(x_1, \ldots, x_r)$ is just a pure transcendental extension of k, rational functions in r variables, so it is easy to embed this subfield into K by just choosing r elements of K that are algebraically independent over k. But now L is an algebraic extension of $k(x_1, \ldots, x_r)$. Since K is algebraically closed, the embedding of $k(x_1, \ldots, x_r)$ into K extends to an embedding of L into K. Voilà, there is our common zero, $k[X_1, \ldots, X_n]/P \subset L \subset K$.

The proof of the serious weak Hilbert Nullstellensatz, that is, with no assumption about K other than that it is algebraically closed, is substantially harder than this wussy result where we assume K has infinite transcendence degree over the prime field. For example, if K is the algebraic closure of k, there is no hope of embedding the field L above into K if the transcendence degree r is greater than 0. It requires much more delicate arguments to find a homomorphism $k[X_1, \ldots, X_n]/P \to K$.