Last time: Motivation to find themes from physics

Today: Actual details of structures:

Definition (TQFT) - Fluid concept - will go through a few steps

(With: Atiyah - Segal - late '70s):

- Several parts: I) Finite $E$ is closed, oriented $(n-1)$-manifolds $\to \mathcal{C}$ vs. if $\to \text{isoms}$

- Physics Note: $E(x)$ is $\mathcal{V}$ of $x$ vs. $\mathcal{V}$ of $x$, the space of fields $\mathcal{V}$

- Example: $x = s^1$ (1-dimensional case) = space of fields $\mathcal{V}$

- $E(s^1) = \Gamma \left( \frac{x}{\mathcal{V}} \right)$.

$E$ satisfies some properties:

- $E(x_1 \sqcup x_2) = E(x_1) \otimes E(x_2)$: For any finite family $x = \bigcup_{k=1}^{n} x_k$.

- For $x = \bigcup_{k=1}^{n} x_k$:

- For $x_1, x_2$, $\mathbb{M} = \mathbb{M}$:

- $E(x_1) \to E(x_2)$

- Multi-linear, satisfying invariance property of such $\mathbb{M}$.

Note: One side $E(x_1) \otimes x_2 \otimes E(x_1)$.

Hence, obtain an $\mathbb{S}$-equiv. map $E(x) \to E(x_1)$.

II). Assign to each $X$ from $x$, to $x$, a linear map $U_Y : E(X_1) \to E(x_2)$ satisfying some properties:

(a) Avoid extra - careful regime for $1^{st}$ cut.

(b) Depends only on different type of $Y$; (fixing $\mathbb{M}$)

(c) $U_Y$.

$dY = \delta x_1, x_2 = \delta x_2 \circ \delta x_1$.
The diagram represents the relation $\Psi_{Y_1,Y_2}^*$ between $Y_1$ and $Y_2$.

Extraneous data: $\Psi_{X,\psi I} = \psi E(X)$,

(Note: Can describe this using 2-category later.)

Note that $d$ is a clone $n$-mild, claim $E(d) = C$:

$E(x) = E(x \oplus \psi) = E(x) \oplus E(d) \Rightarrow E(d) = C$.

We take $d$ to be an $n$-mild view as $\Psi_Y$ from $\Phi$ to $d$.

We obtain $\Psi_Y : C \rightarrow C \Rightarrow \Psi_Y \in C$.

A perspective: numbers assigned to clones $n$-mild, allowing completion and retraction of new operators (rings from pairs).

Question: 1) Do the numbers $\Psi_Y$, $Y$ clone $n$-mild, determine the theory? (if not, what does?)

2) Are there restrictions to the type of deformation invariants that can arise this way?

3) Is $E(x)$, an $n$-mild, spanned by $\Psi_Y$ if $E(x) \mathfrak{Y} x$?

4) Classify TQFTs - moduli space of them? Define invariants of TQFTs.

Exercise (algebraic problem): Let $A$ be a comm. ring $n$-unital, $M$ and $N$ modules over $A$. Then modules are fin. gen., projective modules, and in addition if only if there

JAN 15 2008
Existence of a module $\alpha : A \to \text{Hom}_A(M, N)$, $\beta : \text{Hom}_A(M, N) \to A$. 

1. The following composites are $\beta$:

$$(1) \quad M \xrightarrow{\text{DOM}} M \otimes \text{DOM} \xrightarrow{\text{MOM}} M, \quad N \xrightarrow{\text{NOM}} N \otimes \text{NOM} \xrightarrow{\text{NOMON}} N.$$

**Proof:** In Spanier's topology book (out of space, I state maps).

Inducibly: means $\beta$ detains $\text{MOM}(M, N) - \text{MOM},$ and

Relevance: TFT theory this structure:

$\Psi = X \times I : \text{cobordism from } \mathcal{D} \text{ to } X \sqcup X \text{ - } \mathcal{D}_{\mathcal{D}}.$

So $\alpha = \Psi \xrightarrow{\text{Dom}} \mathcal{D} \Rightarrow E(X) \otimes E(X).$.

Similarly: some cobordism $\psi = X \times I : \text{Hom}_A(M, N) \to \mathcal{D}.$

So $\beta = \phi \xrightarrow{\text{Dom}} E(X) \otimes E(X) \Rightarrow C.$

Why does it satisfy (1)? Couple $(1 \circ \beta) \circ (\Psi \circ \beta)$:

$$\xrightarrow{\text{duality}} \beta : E(X) \xrightarrow{\text{iso}} E(X) \Rightarrow \text{Duality}.$$

What is

inducely rep: $\alpha : E \Rightarrow E(X) \otimes E(X)$ (using $\beta \Rightarrow \text{Dom here}).$

- adjoint to $\beta : E \Rightarrow \text{Hom}_A(E(X), E(X)) = T(E) \otimes \text{Hom}_A(E(X), E(X)).$

- So $\{b_1, \ldots, b_k\}$ a basis for $E(X)$, $(b_1^*, \ldots, b_k^*)$ dual basis,

claim $d(1) = \sum_{i=1}^{k} b_i \otimes b_i^* \Rightarrow$ adjoint of $\beta \Rightarrow \text{Set}$.

13 TFT

JAN 1 5 2008
Similarly: \[ \beta : E(t) \otimes E(x) \to \mathbb{C}, \] given by evaluation (adjoint to \( \beta \)).

Conclude: \( \chi \times S^1 = \dim \mathbb{E}(x) : \text{constant} \chi(r) \), conclude.

Example of a TQFT (Dirac great within the model) - TQFT in dimension 1:

- associated to a finite group \( G \) (also generalizes to continuous groups).
- Fixed: No fermions present.

Fix dimension \( n \):

1. 1st, describe the \( \pi \)-varieties \( Y \) and \( Y \) close under product \( Y \times Y \).

\[ \chi(y) = \text{weight} \ # \ # \text{of \ genus} \ G \text{-bundles over \ } Y. \]

For example, principal \( G \)-bundles \( G \to Y \times Y \) (\( X \) connective).

\[ \text{weight} = \frac{1}{\text{dim} \ G} \quad \text{so} \quad \chi(y) = \chi(1) \quad \text{the identity} \]

\[ \chi(1) = \frac{1}{\chi(Y, K(G))} \quad \text{since \ G \-conjugate}. \]

- Finite \( \chi \).

From covering space theory: \( \chi(Y, \text{simply connected}) \) we know these \( \chi \)-varieties:

- \( G \)-bundles \( Y \times \mathbb{C} \text{quotient} \to \chi \)-varieties \( \chi(Y, K(G)) \text{simply connected} \).

\[ \chi(Y, B) \to \chi(Y, K(G)) \text{simply connected}. \]

- free \( B \)-actions (use \( \chi \)).

If \( \chi \)-p. were finite: \( \chi(Y, B) \neq \chi(Y, K(G)) \).

- Given a rep. \( \rho \to \mathbb{C} \), how do we consider \( \text{Aut}(\rho) \)?

- \( \rho \) arises from a hom \( \rho : Y \to \mathbb{C} \) as follows:

\[ \rho \cong Y \times \mathbb{C} \quad \text{as a \( G \)-module}. \]

\[ \rho : \text{act on \ } Y \times \mathbb{C} \text{on \( Y \times \mathbb{C} \).} \]

- Have residue right \( \mathbb{C} \)-action here now.

Now, \( \rho \)-equivariant maps \( Y \times \mathbb{C} \to Y \times \mathbb{C} \):

- \( \rho \)-acts on \( \mathbb{C} \) on the right.
by 0 - fibration of $\tilde{\mathcal{G}} \to G$, since as $\pi_1$-equiv. $\tilde{\mathcal{G}} \to G$.

Then $(y, 1) \xrightarrow{} (\varphi(y), 0)$, 0 \in \mathcal{G}$, where $\varphi(y) = \rho(x)$ ($\rho: \rho \to \varphi$).

-so can wind, deck & turn:

get map $(y, 1) \to (y', 0)$ - get some $\Theta y \to \Theta y$ associated.

$\Theta$ equivalence:

\[
\begin{array}{ccc}
\mathcal{G} \times \mathcal{G} & \xrightarrow{\Theta} & \mathcal{G} \\
\downarrow \text{act} \rho \times \Theta & & \downarrow \text{act} \Theta \times \rho \\
\mathcal{G} \times \mathcal{G} & \xrightarrow{} & \mathcal{G} \times \mathcal{G}
\end{array}
\]

we sum $\Theta$ to commute w/ $p(x) - \text{twist} \Delta \rho, \mu, \nu$, - cyclic

\[
\begin{array}{c}
\rho(p) \subset \mathcal{G} - \text{then:} \\
\text{Aut}(p) = C(\rho(p(n)))
\end{array}
\]

\[\Theta \in \text{all} \]

Interpretation: $\Theta$ is a clan def. by $\Theta y : \rho : \pi = \nu \in \text{conjug. by} \Theta y$;

so that $\pi \in \frac{|\text{Aut}(p)|}{|\text{Aut}(p)|} = \frac{|\text{Hom}(\pi, \psi(\nu, \phi))|}{|\text{Hom}(\pi, \psi(\nu, \phi))|}$

\[\pi \in \frac{|\text{Aut}(p)|}{|\text{Aut}(p)|} = \frac{|\text{Hom}(\pi, \psi(\nu, \phi))|}{|\text{Hom}(\pi, \psi(\nu, \phi))|} \quad \text{so} \quad \psi(\nu, \phi) = \frac{|\text{Hom}(\pi, \psi(\nu, \phi))|}{|\text{Aut}(p)|}.
\]

\[\text{equiv. to $y$:}

Vector space for $\alpha - \text{a field}, \rho$ maps to $\alpha$-fields:

$E(\alpha) = \mathbb{C}(\rho, \epsilon)$ - is $\alpha$-closure of $\rho$-subalgebra on $\alpha$, all $f \alpha$ from these fields

- $\alpha$-vector space.
Remaining follows from \( \pi_1 \delta \mathfrak{g} \). Consider:

\[
\partial_{y} \phi(x) = x,
\]

take \( \delta_y \phi(x) \in \mathfrak{g} \delta_x \) to be:

\[
\delta_y \phi(x) = \frac{1}{\text{det} \, \mathfrak{g} \delta_x + a},
\]

where \( \mathfrak{g} \delta_x + a \) is \( y \)-indefinite.

Above login of 'path integrals':

- 'integrals w/ 'fields''
- OB - field! \( \mathcal{L} \in \text{aut}(\mathcal{g}) \)

Different perspectives:

\[
\text{given cobordism } \gamma : [x_1, x_2] \to \mathbb{R}^n.
\]

- restriction map:
  \[
  \mathcal{P}_{x_1} \hookrightarrow \mathcal{P}_y \twoheadrightarrow \mathcal{P}_{x_2}
  \]

- apply same only to \( \text{cobordism} \):
  \[
  \mathcal{P}_{x_1} \longrightarrow \mathcal{P}_y \longleftarrow \mathcal{P}_{x_2}
  \]

so the map is \( (\text{post})_0(x, y) \) - not quite definite yet. Transfer w/ shurw map.

for don't work, set that is an claim. \( \gamma \) x x x x.

- \( \mathcal{P}_y : \mathcal{P}_{x_1} \longrightarrow \mathcal{P}_y \) gives by \( \mathcal{P}_y (x) (f) = \chi (\mathcal{P}_y (x), f) \) \( \mathcal{P}_y (x) \)

- \( (\text{post})^1 : \mathcal{P}_y \longrightarrow \mathcal{P}_{x_2} : \text{post}^1 (f) (\mathcal{P}_y (x_2)) = \sum_{f \in \mathcal{P}_y (x_2)} \frac{1}{\text{det} \, \mathfrak{g} \delta_x + a},
\]

\[
\text{cases of }
\text{shurw on } y
\text{while } \mathcal{P}_y (x_2) = \mathcal{P}.
\]

\[
\text{study}
\]

compute how bundles on \( x_1 \) to bundle on \( x_2 \) - or the on such bundle - study bundles on \( y \) restrict to \( x_2 \), thus restrict to \( x_1 \).

16 TFT

JAN 15 2009
Start next categorical framework: 2-categories: $E$

- $\text{obj } E$: if $a,b$ are objects, $\text{Mor}(a,b)$ is a category as well.

- $\text{obj } (\text{Mor}(a,b)) = 1$-morphisms

- $\text{Mor } (\text{Mor}(a,b)) = 2$-morphisms. (composition in $E$)

Ex: start w/ a cobordism category $E$ - step: compactifiability

Ex: (linear) - $\text{obj } E = C$-vector spaces;

- $\text{Mor } (V_1, V_2)$ category:
  - $\text{obj } = \text{linear functors } V_1 \to V_2$;
  - $\text{Mor } (V_1, V_2)$ pairs $\varphi : V_1 \to V_2$

(Forget at a TQFT) - (see Thursday)

Thursday: classify 2-disk theories (from alg. $E$) - Costello.

- algebras of $E$: very non-split into ordinary pieces
  - $\Rightarrow$ implications. (ultimately in $E$ with any use this)