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1/24/08
LN#5

Recall thm from last time:

- A a f.d. comm FA / TQFT, $=F_A$ $\alpha \in A$ distinguished,

$\alpha = F_A(\text{⊚}) : \mathbb{C} \to A$.

- $\{a_i\}_{i=1}^n$ $\mathbb{C}$-basis for A, & a_i^* dual, w.r.t. $\langle , \rangle$, $\alpha = \sum_{i=1}^n a_i a_i^*$.

Thm: A is semisimple (as a module over itself by reg. rep'n) iff α is a unit.

(Reference: Anderson/Fuller, GTM series (for semisimple modules)
Rings & Cats of Modules).

Recall: Let unital a ring R.

Defn: ~~I~~ Suppose T is a left R-module, then T is simple if T has no nontrivial R-submodules. $\Leftrightarrow$ $T \cong R/M$, M maximal (left?) ideal of R.
- wait rather - comm.

- A module M is semisimple if $M \cong \bigoplus_{\alpha \in I} T_\alpha$, T_α's simple

Facts: • TFAE for left R-modules M:

1) M semisimple,
2) every submodule of M is a direct summand,
3) every short exact sequence $0 \to K \to M \to N \to 0$ splits.
(K, N then also s.s.)

Schur's Lemma: if A is a f.d. alg over an alg. closed field k,

and M, N are both irreducible left A-modules, then

1) $\mathrm{Hom}_A(M,N) = 0$ if $M \not\cong N$.
2) $\mathrm{Hom}_A(M,M) \cong k \cdot id_M$.

Cor: If A is commutative, & M irred, M is 1-dim'l : $\dim_k M = 1$.

Pf: mult by $A: M \to M$ is an A-hom by commutativity, so

$A \cdot \alpha = k\alpha. \Rightarrow$

Back to FAs: if A a semisimple comm. FA; then $A \cong \bigoplus_{\beta \in I} \mathbb{C}_\beta$ as A-modules (1-dim'l by cor.)

Facts about SS FAs: · Suppose A an SS. comm FA.

Thm/Lemma: A has a $\mathbb{C}$-basis $\alpha_1', \dots, \alpha_n'$ s.t. $\alpha_i' \cdot \alpha_j' = \begin{cases} 0, & i \neq j \\ \alpha_i', & i = j. \end{cases}$

(idempotent $\mathbb{C}$-basis).

Pf: write $A = \bigoplus_{i=1}^{n} \mathbb{C}\alpha_i$ as A-modules. Note we can choose α_i's can be chosen s.t. $\alpha_i \alpha_j = 0$: pick α a generator of an irred submodule $\mathbb{C}$; consider map $A \to \mathbb{C}_\alpha$, $a \mapsto a \cdot \alpha$;

- kernel is a direct summand and is a SS FA (Ex why?)

  - continue until exhausted (finite dim'n).

- Let $\{\beta_1, \dots, \beta_n\}$ be dual to these α_i – $\langle \alpha_i, \beta_j \rangle = \delta_{ij}. = \theta(\alpha_i \beta_j)$.

Write $\beta_i = \sum_{k=1}^{n} z_{ik}\alpha_k$, so $\beta_i \alpha_j = z_{ij}\alpha_j^2$. Hence, $z_{ij}\,\theta(\alpha_j^2) = \delta_{ij}$.

so $\theta(\alpha_j^2) \neq 0$; claim $\alpha_j^2 = c_j\alpha_j$, $c_j \in \mathbb{C} \setminus 0$.

- define $\alpha_j' = \frac{1}{c_j}\alpha_j$ – forces idempotence.

- rewrite now as $\{\alpha_j\}$ – no primes. ; let $\beta_1, \dots, \beta_n$ be dual basis:

so $\beta_i = \sum_j k_{ij}\alpha_j$; $\beta_i \alpha_j = k_{ij}\alpha_j^2 = k_{ij}\alpha_j$.

$\delta_{ij} = \theta(\beta_i \alpha_j) = k_{ij}\,\theta(\alpha_j)$. Hence, if $i = j$, $\theta(\alpha_j) = \frac{1}{k_{jj}} \neq 0$.

$i \neq j$: $0 = k_{ij}\,\theta(\alpha_j)$; $\theta(\alpha_j) \neq 0$, so $k_{ij} = 0$.

Hence $\beta_j = k_{jj} \alpha_j$, k_j det by $\Theta(\alpha_j)$.

Shows Cor: ~~f.d.~~ n-dim'l comm SS FA's over $\mathbb{C}$ are classified by n nonzero complex numbers $z_1, \dots, z_n$ taking the values $z_i = \Theta(\alpha_i)$.

- so ~~[illegible]~~, such 2 FA's are isomorphic if they have the same list of #'s.

CQ. about characterizing $\mathbb{C}[A_{\mathcal{B}}\, \varphi_p]$ this way).

Lemma: Let A be a f.d. comm. alg over an alg closed field k. Suppose $M \subset A$ ~~is~~ an A sub module is irreducible (M a simple ideal of A). if M is not a field, then it has nilpotent elts.

PP: Exercise.

Back to Thm: Suppose A is an f.d comm FA, semisimple over $\mathbb{C}$. WTS φ a unit. write $A = \bigoplus_{i=1}^{n} \mathbb{C}\alpha_i$ as before. w/ α_i, β_i as before.

recall: distinguished elt $\varphi = \sum_{i=1}^{n} \alpha_i \beta_i = \sum_{i=1}^{n} k_i \alpha_i$, $\Theta(\alpha_i) = \frac{1}{k_i}$.

so $\varphi = \sum_{i=1}^{n} \frac{\alpha_i}{\Theta(\alpha_i)}$; claim $\varphi^{-1} = \sum_{i=1}^{n} k_i^{-1}\alpha_i$. (need $1 = \sum_{i=1}^{n} \alpha_i$) fact. - check

- check $\varphi \cdot \varphi^{-1} = \underline{\sum_{i=1}^{n} \alpha_i}$ - clearly $=1$.

Opposite direction: Now assume φ is a unit. WTS A is S.S.

we'll show there are no nonzero nilpotent elts of A.

(exer: this characterizes SS algs by lemma contrapositive).

Let $\eta \subset A$ be the ideal of nilpotents in A. we show $\varphi \cdot \eta = 0$, (so $\eta = 0$).:

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Filter A as follows: let $S_1 = \mathrm{ann}(\eta) \subset A$ $(= \{a \in A : a\eta = 0\})$.

so not to show $\eta \in S_1$. Let. $\pi_i : A \to A/S_{i-1}$, ~~& let $S_i = \pi_i^{-1}($~~

Let $S_i = \pi^{-1}(\mathrm{Ann}(\mathrm{Nil}(A/S_{i-1})))$

$= \{a \in A : \text{for } x \text{ . } \in \text{ rad } S_{i-1},\ \underline{ax \in S_{i-1}}\}$.

gives a filtration $S_1 \subset S_2 \subset \cdots \subset S_k = A$ on A. ; pick basis for

A: pick for S_1, extend to basis for S_2 & so forth. $\to$ basis of A.

denote basis by $\{e_1, \ldots, e_n\}$. w/ duals $\{e_i^*\}$. Suppose $e_i \in S_j \setminus S_{j-1}$,

and $a \in \eta$. Then $a \cdot e_i$ is also nilpotent, (S_j comm.) so

$ae_i \in S_{\underline{j-1}}$ (in fact, $\underline{S_1}$) so can be expressed as an LC.

of e_k's, $\underline{k<i}$. Hence, $\langle ae_j, e_i^* \rangle = 0$.