M283
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LN #7

Recall: "meta"-thm about TCFTs: Class cx $E(S^1)$ is an algebra; in fact, a BV-algebra.

- Algebraic construction w/ geometric context:
- Hochschild (co)homology – (refs: Loday-Quillen paper on cyclic homology; Loday's Cyclic Homology; + other refs)

A an assoc alg/k (field) (more generally, works on A_∞-algs)

define the Hochschild chain cx $CH_*(A)$: $\to A^{\otimes(n+1)} \xrightarrow{b} A^{\otimes n} \xrightarrow{b} \cdots \to A$,

where $b(a_0,\dots,a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0,\dots,a_i a_{i+1},\dots,a_n)$
$+ (-1)^n (a_n a_0,\dots,a_{n-1})$. – turns a differential, making this a cx.

$H_*(CH_*(A))$ called the Hochschild homology of A ($HH_*(A)$).

– don't tie ones'self down to one cx:

consider acyclic cx (acyclic resolution of A as a left $A\otimes A^{op}$-module $= A^e$)

: $n \mapsto A^{\otimes(n+2)} \xrightarrow{b'} A^{\otimes(n+1)}$, $\cdots \to A^{\otimes 2} \xrightarrow{(mult)} A$; b' given by:

$$b'(a_0,\dots,a_{n+1}) = \sum_{i=0}^{n} (-1)^i (a_0,\dots, a_i a_{i+1},\dots,a_n).$$

The $A^{\otimes(n+2)}$ are free A^e-modules (assuming some flatness condition on $A \to k$).

- acyclic: has a null homotopy: $A^{\otimes n} \xrightarrow{s} A^{\otimes(n+1)}$ taking $(a_0,\dots,a_{n-1})$ to $(1,a_0,\dots,a_{n-1})$.

– see $b's + sb' = id$, so $id \simeq 0$.

Consider A as a right $A\otimes A^{op}$-module; ($\alpha\cdot(a_1\otimes a_2) = a_2\,\alpha\, a_1$) then consider

$A \otimes_{A\otimes A^{op}} A^{\otimes(n+2)} \xrightarrow[\cong]{\varphi} A^{\otimes n+1}$; in fact, φ gives chain cx isom from $A\otimes_{A\otimes A^{op}} \tilde{C}_* \to CH_*(A)$.

$\alpha\otimes(a_0,\dots,a_{n+1}) \mapsto (a_{n+1}\,\alpha\, a_0, a_1,\dots,a_n)$.

Hence $HH_*(A) = \mathrm{Tor}^{A\otimes A^{op}}(A,A)$ (homological defn).

Variations on this for different types of algebras / modules

- M an A-A-bimodule (ie, right $A\otimes A^{op}$-module);

define $HH_*(A,M) = \mathrm{Tor}_*^{A\otimes A^{op}}(M, A)$ (Ralph says $\mathrm{Tor}^{A\otimes A^{op}}(A,M)$)

$= H_*(\tilde{C}_* \otimes_{A\otimes A^{op}} M)$

– ie, homology of cx $M\otimes A^{\otimes n} \to M\otimes A^{\otimes(n-1)} \to \cdots$

$$b(m, a_1, \ldots, a_n) = (ma_1, a_2, \ldots, a_n) + \sum (-1)^i (m, a_1, \ldots, a_i a_{i+1}, \ldots) + (-1)^n (a_n m, a_1, \ldots, a_{n-1}).$$

Cohomology version: $HH^*(A,M) = \mathrm{Ext}^*_{A\otimes A^{op}}(A,M)$, where $= H_*(CH^*(A,M))$,

$CH^*(A,M)$ similarly

$$\cdots \to \mathrm{Hom}(A^{\otimes n-1}, M) \xrightarrow{\delta} \mathrm{Hom}(A^{\otimes n}, M) \to \cdots$$

where $(\delta\phi)(a_1, \ldots, a_n) = a_1 \phi(a_2, \ldots, a_n) + \sum_{i=1}^{n-1} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots) + (-1)^n \phi(a_1, \ldots, a_{n-1}) a_n$

(note: 'sensible' differentials come from underlying simplicial structure)

Connes' cyclic homology:

consider $CH_*(A)$: $(\to A^{\otimes n+1} \to A^{\otimes n} \to \cdots)$

- $CH_n(A)$ has action of $\mathbb{Z}/n+1 = \langle t \rangle$: (ie, is $k[\mathbb{Z}/n+1]$-module):

$t(a_0, \ldots, a_n) = (-1)^{nn} (a_n, a_0, \ldots, a_{n-1})$.

Suppose A is an algebra over a field k of char 0. Claim:

Mod out by $\mathbb{Z}/n$-actions: get new cx: (where $M//\mathbb{Z}/n = M \otimes_{k[\mathbb{Z}/n]} k$) "coinvariants"

$$\to A^{\otimes n+1}//\mathbb{Z}/n+1 \xrightarrow{b} A^{\otimes n}//\mathbb{Z}/n \to \cdots$$

$\to H_*$ of this is cyclic homology; cx.

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dual setup: cyclic cohomology (analogue of diff. forms):

$n \mapsto \mathrm{Hom}_{k[\mathbb{Z}/n]}(A^{\otimes n}, k)$ where $k[\mathbb{Z}/n]$ acts trivially on k $\searrow$ $\mathrm{Hom}_{k[\mathbb{Z}/n+2]}(A^{\otimes n+2}, k) \to \cdots$ invariants of the $\mathrm{Hom}(A^{\otimes n+2}, M)$'s.

- cyclic-gp invariance $\to$ $CH^k(A)$. $\to$ non-comm. geometry theories

Loday-Quillen: algebras over arbitrary fields;

too brutal to mod out; take homotopy orbits instead:

resolve k over $k[\mathbb{Z}/n]$, take double complex:

(in std. elem cyclic gp resoln) $\quad (N_k = \sum_{i=0}^{k-1} t_k^i)$, $N(1-t)=0$ & $(1-t)N=0$

(1st step: horiz arrows from cyclic resolution.)

$$
\begin{array}{llccccc}
 & & \vdots & & & & \\
 & & \downarrow b & & \downarrow b' & & \downarrow b \\
 & 2 & A^{\otimes 3} & \xleftarrow{1-t_3} & A^{\otimes 3} & \xleftarrow{N_3} & A^{\otimes 3} \\
 & & \downarrow b & & 1-t_2 (\downarrow) b' & & (\downarrow) b \\
\mathbb{Z}/2 & 1 & A\otimes A & \leftarrow & A^{\otimes 2} & \xleftarrow{N_2} & A^{\otimes 2} \\
 & & \downarrow b & & (\downarrow) b' & & (\downarrow) b \\
\mathbb{Z}/1 & 0 & A & \xleftarrow{1} & A & \xleftarrow{1} & A \\
 & & & & & & \\
 & & 0 & & 1 & &
\end{array}
$$

(H_k of horiz lines computes $\mathrm{Tor}_{k[\mathbb{Z}/n]}(k, A^{\otimes n})$ $= H_*(\mathbb{Z}/n, A^{\otimes n})$ (gp. cohom).)

odd vertical columns: inject b' differential.

- so odd columns are acyclic cx's $\to$ double complex $CC_*(A)$. (check squares anticommute)

L-Q: define $HC_*(A) = H_*(\text{double cx})$ $(= H_*(\mathrm{Tot}(\text{double cx})))$.

- by filtration by rows: $\exists$ S.S.

$\bigoplus_n H_*(\mathbb{Z}/n; A^{\otimes n}) \Rightarrow HC_*(A)$

- positive-degree $H_k(\mathbb{Z}/n+1, -)$ gps all 0 if $\mathrm{char}\, k = 0$ so

this reduces to Connes's chain cx in char 0.:

$$\left(H_p(\mathbb{Z}/n, A^{\otimes n}) = \begin{cases} 0, & p>0 \\ A^{\otimes n}/(\mathbb{Z}/n), & p=0 \end{cases} \right)$$

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Hence L-Q gives a generalization of Connes's construction.

- convert cyclic gps to an S^1-action: finite subgps of $S^1 \subset \mathbb{C}$.

1st: (Aside: Connes's $THH(A)^{S^1}$ - related.)

recall Δ-operator in BV-algebra defn: - produce degree-lap in $HH_*(A)$:

- Let $B: A^{\otimes n+1} \to A^{\otimes n+1}$ be given by:

$$A^{\otimes n+1} \xrightarrow{N} A^{\otimes n} \xrightarrow{s} A^{\otimes n+1} \xrightarrow{1-t} A^{\otimes n+1}; \quad \text{note } B^2 = 0$$
(odd) s/c $N(1-t) = 0$.

L-Q: B simplifies double cx: kill acyclic columns at cx

reduce to:

$$\begin{array}{ccc} A^{\otimes 3} & \nwarrow B & A^{\otimes 3} \\ \downarrow b & & \downarrow b \\ A^{\otimes 2} & & A^{\otimes 2} \\ \downarrow b \nwarrow B & & \downarrow b \\ A & & A \\ 0 & & 2 \end{array} \quad \cdots$$

$= CH_*(A) \otimes k[c]$ $|c| = 2$.
($\cong H_*(\mathbb{C}P^\infty; k)$.
BS^1

$\to$ get a double cx again (may deal w/ sign classes).

✓ check gives correct diff

Let $\mathcal{B} = (CH_* A \otimes k[c], B + b)$.
Tot

claim: $\mathcal{B}$ is quasiisomorphic to $CC_*(A)$; clear map

$\mathcal{B}^{Tot} \to CC_*(A)$ $\in CC_*(A)_{p-1,q+1}$ (check quasi isom)
- use s to kill acyclic columns).

$x_{p,q} \mapsto (x, sNx)$

Then $HC^*(k) = k[c] = H^*(BS^1, k)$. (as k-modules)

- will be an algebra when A commutative.

- $HC_*(T(V)) = \bigoplus_n H_*(\mathbb{Z}/n; V^{\otimes n})$.
$\parallel$
$(\bigoplus_{n \geq 0} V^{\otimes n})$

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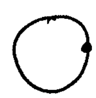
Role of S^1 / cyclic groups here:

•  has a simplicial description: 1 0-simplex, 1 1-simplex (nondegenerate) but need degeneracies of simplices:

in a s. set $X_\bullet$, have sets X_n, face maps $d_i : X_n \to X_{n-1}$, $i = 0, \dots, n$, & degen maps $s_i : X_n \to X_{n+1}$, $i = 0, \dots, n$, w/ simplicial relations.

Then can form geom. realization $|X_\bullet| = \bigcup \Delta^k \times X_k \Big/ \{ (\delta^i t, x) \sim (t, d_i x), \ (\sigma^i t, x) \sim (t, s_i x) \}$

(where the δ^i, σ^i make Δ^* a cosimplicial space)

– omit formulas – inclusions (fixed collapsing): $\delta^i : \Delta^n \to \Delta^{n+1}$ ith face

$\sigma^i : \Delta^{n+1} \to \Delta^n$ – collapse onto jth face.

– or 'along' j–$j+1$ edge.

S-set from nondegen simplices – add degens:

• define corr. simplicial set S^1_{sk}: $S^1_0 = \{0\}$; $S_1 = \{ s_0 \overset{0}{*}, 1 \}$ $\overset{\text{degen.}}{0=}$

$S_2 = \{ 0', s_0 1, s_1 1 \}$ ← all degenerate. – get $n+1$ elts in S_n from combinatorics of s_i's & relations.

• face maps: $d_i : S_n \to S_{n-1}$ $\quad d_i(k) = \begin{cases} k, & k \le i \\ k-1, & k > i \end{cases}$;

$s_i(k) = \begin{cases} k, & k \le i \\ k+1, & k > i \end{cases}$

Then $|S_\bullet| \cong S^1$. (homeomorphic – killed degen 2-simplices & higher.)

As a set $S_n \cong \mathbb{Z}/n+1$. – no group struct (yet).

Now: suppose we want to study LX, (X conn) $= Map(S^1, X)$.

$\underset{\text{homeo}}{\cong} Map(|S_\bullet|, X) =: Map\left(\coprod_k \Delta^k \times S_k / \sim, X\right)$ finite pows.

$\underset{\text{(constraints) to follow}}{\subseteq} Map\left(\coprod_k \Delta^k \times S_k, X\right) = \prod_k Map(\Delta^k, X^{S_k} \cong X^{k+1})$

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constraints in $\prod \mathrm{Map}(\Delta^k, X^{k+1})$: X^{S_k} gives cosimplicial space :

cofaces — where structural maps are:

(repeat) $X^k \xrightarrow{d_i^k} X^{k+1}$: $(x_0, \ldots, x_k) \mapsto (x_0, \ldots, x_i, x_i, \ldots x_k)$

(delete) $X^k \xrightarrow{s_i^k} X$ (codegen) — (in fact cocyclic space?) $\to S^1$ action?

image of LX = sequences (F_k) respecting / compatible w/ coface/codegen maps

$= \mathrm{Map}_{\mathrm{Top}^\Delta}(\Delta^*, \underline{X}^{S_k})$ (has top space?) — called the totalization of X^{S_k}. $:= \mathrm{Tot}(X^{S_k})$.

for each k, taking adjoints, have $\Delta^k \times LX \xrightarrow{f_k} X^{k+1}$; (very explicit:)

— eval loop γ on (non)barycentric coords of Δ^k: — endpoint.

$\Delta^k = (0 \le t_1 \le \cdots \le t_k \le 1) \times \gamma \mapsto (\gamma(t_0), \ldots \gamma(1)) \in$ each side of cosimplicial

compute $H_k(LX)$: gives map $C_k(LX) \to C_{k+k}(X^{k+1})$

(EZ map) $\to$ $\downarrow id_{\Delta^k} \times LX$ $\nearrow C_{k+k}(d_k)$

$C_{k+k}(\Delta^k \times LX)$

(EZ) $C_k(\Delta^k) \otimes C_*(LX) \xrightarrow{EZ} C_{k+m}(\Delta^k \times LX)$.

dualize: $(C^*(X))^{\otimes k+1} \to C^*(X^{k+1}) \to C^{*-k}(LX)$

$\partial_i^k = \Delta$.

claim: (John Jones, '87) cosimp maps induce cup products — mult in $C^*(X)$

$\to$ b-operator in Hochschild homology; — chain maps.

$$\begin{array}{ccc} C_*(X)^{\otimes k+1} & \xrightarrow{d_k} & C^{*-k}(LX) \\ \downarrow b(\text{+int}) & & \downarrow \delta \\ C_*(X)^{\otimes k} & \xrightarrow{d_{k-1}} & C^{*-k+1}(LX) \end{array}$$

convergence condition (Anderson): — $H^* \mathrm{Map}_\Delta(\Delta^0, X^{S_0})$; Jones: — own — fine; intuitively. if X 1-conn, LHS complete $CH_*(C^*X) \to C^*(LX)$ is a chain equivalence.

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if X 1-conn:

Cor: $HH_*(C^*X) \cong H_* LX$. ($H^* LX$?)

- really, $LX \simeq$ subspace $Map(\Delta^n, X^{n+1})$

Now: X a mfld; cochains have FA upto chain htpy
$\Leftrightarrow$ FA up to htpy $\to$ C-S product. Schwarz top. S

- (ok for $C^{*}K \simeq C^*X$)

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