M283
2/7/08
LN# 9

Back to TQFTs, TCFTs, etc.:

Geometric interpretation of this algebraic material.

Recall: • a Gerstenhaber algebra (or a braid algebra) is a pair $\{B, \{,\}\}$, B a gr-comm algebra, $\{,\}$ is a Lie bracket satisfying the Jacobi identity, s.t. $\{,\}$ is a derivation in each variable. (w/ proper grading assumptions)

• a BV algebra is an example of a braid algebra: $\{A, \Delta\}$ A commutative, Δ has deg 1, $\Delta^2=0$, st.

$[a,b] = (-1)^{|a|}\Delta(ab) + (-1)^{|a|}\Delta(a)\cdot b - a\cdot\Delta b$ then makes A a braid algebra.

Results: Thm (Gerst, '63) $HH^*(A,A)$ is a Gerst. alg (braid alg). (if A is an assoc alg) (pf sketch) w/ cup prod + extra prod.

Thm • (notation) if A is Frobenius, $H^*(A,A)$ is a BV-alg;

• (Costello) $HH^*(A,A)$ is a 2-dim'l TCFT. (i.e. is u.s. for TCFT) (w/ certain univ. property)

– will prove a 2D TCFT is a BV alg.

Thm: A genus-0 2-dim'l TCFT is the same as a BV-alg.

(equivalent categories?)

genus-0: only morphisms = surfaces of genus 0 – (don't get higher genus thru composition?)

PF (Gerst): 'old-fashioned' constructions – cup-i products (Steenrod) – pre-date Steenrod algebra, really.

recall $C^{*} = CH^{*}(A,A) = \bigoplus_{n=0}^{\infty} \mathrm{Hom}(A^{\otimes n}, A)$ -

- cup ($\cup$) - product: $C^p \otimes C^q \to C^{p+q}$ given by

$$(c_1 \cup c_2)(a_1, \dots, a_p, a_{p+1}, \dots, a_{p+q}) = c_1(a_1, \dots, a_p) \cdot c_2(a_{p+1}, \dots, a_{p+q}).$$

cup-1-product: $C^p \otimes C^q \xrightarrow{-\cup_1-} C^{p+q-1}$. ($A$ a DGA) - track internal degrees

$$(c_1 \cup_1 c_2)(a_1, \dots, a_{p+q-1}) := \sum_{i=0}^{p} (-1)^{(|c_1|-1)(|a_1|+\dots+|a_i|-i)} c_1(a_1, \dots, a_{i-1}, c_2(a_i, \dots, a_{q+i-1}), a_{q+i}, \dots, a_{p+q-1}),$$

(cup operation).

- even higher cup-i - products in more commutative cases

$[c_1, c_2] = c_1 \cup_1 c_2 - (-1)^{(|c_1|-1)(|c_2|-1)} c_2 \cup_1 c_1$. - like Lie alg. bracket for endomorphisms.

- check Jacobi + derivation identities

. braid algs & BV-algs are "algebras over specific operads".

relevance of operads to field theory: if C^* is a TCFT, recall have homs

$C_*(M_{n,1}) \otimes (C^*)^{\otimes n} \xrightarrow{\theta_n} C^*_*$. - operations are somehow compatible as n varies

$\rightsquigarrow$ operad structure. - encodes/parametrises mult. operations $n \to 1$.

ex: associative operad governs gp. mult, etc.

- up to htpy? different (A_∞) operad. - space of mults.

have recall P: , $\mathcal{D}$ gives extra structure on TCFT;

TCFT has more operations.

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<u>Operads</u>: (space-valued, etc. — cat-valued operads, (Lurie))

• Let $\mathcal{C}$ ($\square$) be an SMC; (ex: ($\underline{\text{Top}}$, $\times$), (Vect_k, $\otimes$), (chcx(k), $\otimes$), (Cats, $\times$)).

An S-<u>Module</u> a in $\mathcal{C}$ is a sequence of objects $k \mapsto a(k)$ which are <u>representations</u> of S_k. ($S_k \to \mathcal{C}(a_k, a_k)$ monoid maps?).

• an <u>operad</u> a in $\mathcal{C}$ is an S-module, together with <u>maps</u>

$\xi_k \in \mathcal{C}(a(k) \square a(j_1) \square \cdots \square a(j_k), a(\sum_i j_i))$ w/ compatibility <u>props</u>:

& $1 \in a(1)$ ($= \mathcal{C}(I, a(1))$?

①. $a(l) \square (a(m_1) \square a(n_{1,1}) \square \cdots \square a(n_{1,m_1})) \square \cdots \square (a(m_l) \square a(n_{l,1}) \square \cdots \square a(n_{l,m_l}))$

$$\longrightarrow a(\sum_{i=1}^{l} m_i) \square a(n_{1,1}) \square \cdots a(n_{l,m_l})$$

$$\begin{array}{ccc} & \circlearrowleft & \\ \downarrow {\scriptstyle 1 \square \xi_{m_1} \square \cdots \square \xi_{m_l}} & & \downarrow \xi_m \\ a(l) \square a(\sum_{j=1}^{m_1} n_{1,j}) \square \cdots \square a(\sum_{j} n_{l,j}) & \xrightarrow{\xi_l} & a(\sum_{i=1}^{l} \sum_{j=1}^{m_i} n_{i,j}). \end{array}$$

– for <u>assoc</u> <u>alg</u>: each $a(l)$ is $\underline{S_l}$.

② <u>unit</u> +: <u>respects</u> S_n-action (equivariant); $\xi_1(1)$:

$$I \square a(h) \to a(1) \square a(h) \xrightarrow{\xi_1} a(h) = \cong a(h), \forall h.$$

∘ Invented by Boardman-Vogt; term due to May (GOILS)

– used in physics a lot too.

<u>Ex</u>: given a PROP, $\mathcal{C}$, have an <u>operad</u> by restriction: w/.

$\bar{\mathcal{C}}(k) = \underline{\text{Mor}_{\mathcal{C}}(k, 1)}$; struct maps in $\bar{\mathcal{C}}$:

$M_w(k,1) \times (M_w(n_1,1) \times \cdots M_w(n_k,1)) \to M_w(\sum n_i, 1)$: gluing :

$\downarrow_{\Delta \times} \ast \Box$ in prop. $\quad \circ \nearrow$ comp

$M_w(\sum_i n_i, \sum_i 1 = k)$

$Y \circ (Y, 1)$

$= Y \circ (Y \sqcup 1) = Y$.

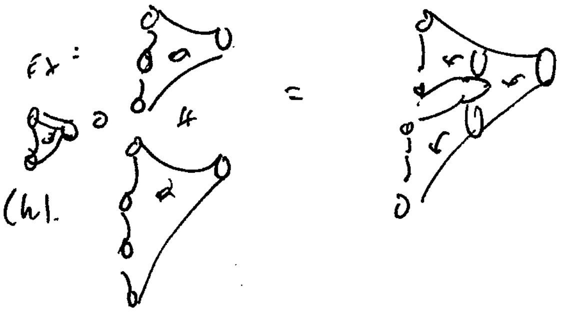
recall? are PROPs: (examples:)

① Segal prop M ② Deligne Segal prop [redacted] : one object for each diff type $2 \to 1$.

③ $End(V)$ - prop..

①: $M_w = M(n,m)$;

Ex:



- topologically as

③: $Hom_k(V^{\otimes k}, V). = End(V)(k)$.

V in any SMC obj.

- over top, say.

An algebra X over an operad A is a morphism of operads in $\mathcal{C}$.

$a \xrightarrow{f} End\, X$. - i.e. for each k, get a map

$a(k) \to \mathcal{C}(X^{\otimes k}, X) \overset{adj}{\leftrightarrow} a(k) \otimes X^{\otimes k} \to X$. - if $\mathcal{C}$, $\times$ is Top,

have $\exists$

have got at $a(k) \times_{S_k} X^k \to X$; if $a(k)$ connected,

S_n-action: have got at

- say $k=2$,; $a(2) \times X^{\otimes 2}$ - action $\sigma = (12)$ - new map; but $= $ htpy since 2 products $\to$ htpy comm

2 paths show.

as

if $a(n)$ a pt: get comm operad. $\to$ alg / pt = assoc, comm monoid in $\mathcal{C}$.

$a(n) \simeq k \Rightarrow E_\infty$-spaces - htpy comm (up to ...) ... .

Goal: define operads in $Ch(k)$ / gr vect ... b, bv

s.t. algebras over b, bv are Gerst., / BV-algs.

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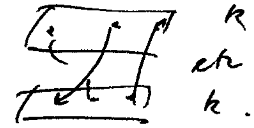
let B_k = braid gp (E. Artin, ~1920s) on k strings. –



isotopy classes of such:

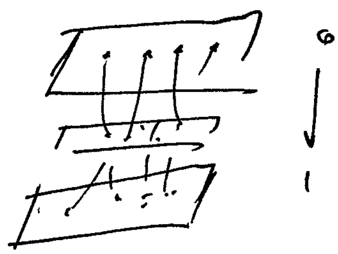
– presentation?

$F(M,k) = \{(x_1,\dots,x_k) \in M^k \text{ distinct}\}$. – S_k acts freely on $F(M,k)$;

$C(M,k) = F(M,k)/S_k$. – k unordered dist. pts.

~~claim: $C(\mathbb{R}^2,k)$~~ Defn: $\pi_1(C(\mathbb{R}^2,k)) = B_k$. :

picture sp. · k pts



– order at top does not corr. to order at bottom:

or: have cover $S_k \to F(\mathbb{R}^2,k) \to C(\mathbb{R}^2,k)$.

note $\pi_1 F(\mathbb{R}^2,k) = PB_k$ – pure braids; $PB_k \hookrightarrow B_k \to S_k$ from π_k – LES.

claim: $F(\mathbb{R}^2,k)$, $C(\mathbb{R}^2,k)$ are $K(\pi,1)$'s;

pf: $F(\mathbb{R}^2,k) \to F(\mathbb{R}^2,k-1)$ fibration; fibre is $\mathbb{R}^2 \setminus \{k-1 \text{ pts}\} \simeq \vee^{k-1} S^1 \simeq K(\pi,1)$.

$(x_1,\dots,x_k) \to (x_1,\dots,x_{k-1})$

& $F(\mathbb{R}^2,1) = \mathbb{R}^2 \simeq *$.

now: same arg for $C(\mathbb{R}^2,k)$. – $F(-)$ covers it.

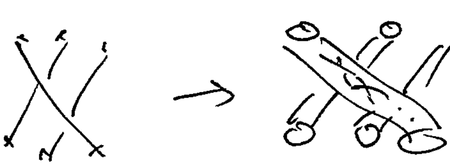
~~back to field theories~~:

One last gp: ribbon braid gp: P_k (pants). $= \mathbb{Z} \wr B_k$

$= B_k \ltimes \mathbb{Z}^k$: action on $\mathbb{Z}^k$: $B_k \to S_k$ – permute factors.

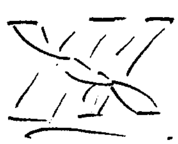
– track twists on strands.

– $< B_{2k}$:



– are k bands, of ribbons.

or



, etc.

defn: $P_k = \pi_1(\text{Gen Rat}_k)$ ↙ rat'l fns $\frac{p}{q}$ ~ monic, deg k, reduced.

[Milgram–Cohen +2] — holom map $\mathbb{P}^1 \to \mathbb{P}^1$ has deg k

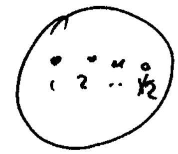
— generic: has simple poles

finally, back to field theories:

cent set finite

Thm (Smale) $PB_k = \pi_0(\text{Diff}(D^2_k, \{x_1, \dots, x_k\}, \partial D^2))$ pointwise

$B_k = \pi_0(\text{Diff}(D^2_k, \{x_1, \dots, x_k\}, \partial D^2))$ setwise pres.



$P_n = \pi_0(\text{Diff}(\text{disk}, \{ooo\}, O\})$ setwise pres., pointwise

$\tilde{P}_n = \pi_0(\text{Diff}(\,\cdot\,, \partial))$ ← also app. in field theory.

each diff. sp has contractible components : so htpy - discrete.

have $K(\pi,1)$'s for these

⟷ models for $B\,\text{Diff}$.

Smale Thm: $\text{Diff}^+(D^2, \partial) \simeq *$. → (gens · 0) = models ~~...~~ of. ∧ config spaces.

M_k → operad (b for B_n, ~~...~~ bw for P_n.)