M 283
2/12/08
W#10

[Lec. notes by Greg Moore - physicist; has paper w/ Segal - notes cute - cool pictures!]

Configuration Spaces : $F(\mathbb{R}^2, k)$ & $C(\mathbb{R}^2, k)$ - are $K(\pi,1)$'s for (pure) braid gps.

origin: loop space operads ('70s, '80s)

1st: note $F(\mathbb{R}^2, k) = \mathrm{Emb}([k], \mathbb{R}^2)$, $[k]$ a 0-mfld | more modern viewpt.
$C(\mathbb{R}^2, k) = \mathrm{Emb}([k], \mathbb{R}^2)/\mathrm{Diff}([k])$.

- take tubular nbhd of disks  want pts in D^2.
  - lots of choices - which ones? perh. cut-off disks choice?

→ leads to the little - cubes operad: (Boardman - Vogt, 1970; May, 1972).

$\mathcal{C}(k) = \mathrm{Emb}\left(\coprod_k D_i^2, D^2\right)$, which when restricted to each given D_i^2, is a composition of translations + dilations of a std D^2.

Note $\mathcal{C}(k) \underset{\Sigma_k}{\simeq} F(\mathbb{R}^2, k)$. - by taking center pt; inverse map - pick ε min distance ctsly, take $\varepsilon/4$ - radius disk around each pt.

- $\{\mathcal{C}(k)\}$ forms an operad in Top.

Let $P(k)$ be the framed little - disks operad: same idea, but, now $P(k) \subseteq \mathrm{Emb}\left(\coprod_k D_i^2, D^2\right)$ - allowing rotations as well!

$P(k) \simeq F(\mathbb{R}^2, k) \times (S^1)^k$ ← track twist parameter / rotation at disk ($S^1 = SO(2)$)

- (More generally, for little n-spheres, should have $SO(n)$ copies).

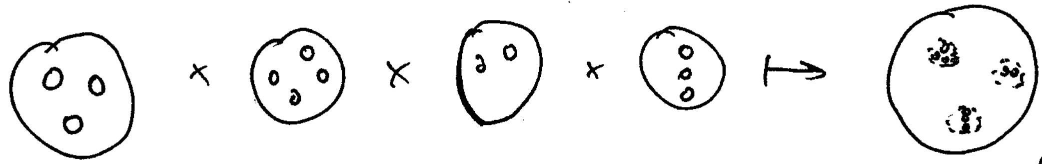
Note: $\pi_1(F(\mathbb{R}^2,k)/\Sigma_k) = B_k$, $\pi_1(P(k)) = P_k$ (vs. 3 section sp.)
RB_k.

(important) **Thm:** 1) A braid algebra is the same data as an algebra over $H_*(\mathcal{C}(k))$. (in graded vector spaces);

2) A $\overline{BV \text{ alg}}$ is the same as an alg over $H_*(P(k))$..

~~Note~~: operad struct: $\mathcal{C}_2(k) \times \mathcal{C}_2(j_1) \times \cdots \times \mathcal{C}_2(j_k) \to \mathcal{C}_2(\sum_i j_i)$:

$k=3$:



– glue in larger disk according to $\times$ form on D^2_i-discs.

Consider $\pi_2 X = [(S^2,\omega),(X,x)]$ w/ op struct:

$= [(I^2, \partial I^2), (X, x)]$ - ~~[illegible]~~ by $\boxed{\begin{array}{c}\alpha\\ \beta\end{array}} \to X$

– $\mathcal{C}_2$ arose from considering all ways to compose two cubes;

– this is May's version, certainly.

– so have $\mathcal{C}_2(k) \times_{\Sigma_k} (\Omega^2 X)^k \to \Omega^2 X$ map – really a ... $\in$, $(\alpha_1,\alpha_2,\alpha_3)$ Pontr. – Then Consider:

– collapse 2-disc on bdry:

$/\partial D^2 \xrightarrow[\text{(///) area}]{\text{coll.}} \bigvee_3 S^2_i \xrightarrow{\alpha_1 \vee \alpha_2 \vee \alpha_3} X = \underline{g(c;\alpha_1,\alpha_2,\alpha_3)}$

Thom space at TN of embedded pts.

so $\Omega^2 X$ is a $\mathcal{C}_2$-algebra;

& $H_*(\Omega^2 X)$ is a braid alg (by Thm) since $H_*(\mathcal{C}_2)$-alg.

free algebra on an operad: Y a (based) space; free $\mathcal{C}_2$ operad

$\mathcal{C}_2(Y) = \coprod \mathcal{C}_2(k) \times_{\Sigma_k} Y^k / (c, y_1, \ldots, *, \ldots, y_k) = (\partial_i c, y_1, \ldots, \hat{*}, \ldots, y_k)$.

insert $1 \to \mathcal{C}_2(1)$ (remove ith cube).

– has univ. prop;

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free 2-fold loopspace gen'd by Y: 1st: univ prop of free alg:

- $Y \subset C_2(Y)$, $Y \to a$; $\exists!$ C_2-alg map $C_2(Y) \to a$, a a C_2-alg.

– $C_2(Y)$ generates free monoid on $(Y, *)$.

free 2-fold loopspace on Y: $Y \to \Omega^2 Z$; $\Omega^2 F_Y \xrightarrow{\exists!} \Omega^2 Z$ ← map of Ω^2-fold loopspaces (i.e. C_2-alg map?)

– so corresp (i.e.) from cts map $F_Y \to Z$.

– adjoint: for any space, set $\Sigma^2 Y \to F_Y$, $\Sigma^2 Y \to Z$, $\exists!$ $F_Y \to Z$ – has extension property $\Rightarrow F_Y \cong \Sigma^2 Y$. (by universality)

– so free 2-fold loopspace on Y is $\Omega^2\Sigma^2 Y$.

Thm (May; Segal, Milgram) for Y connected, $C_2(Y) \xrightarrow[\text{collapse}]{\text{p.T}} \Omega^2\Sigma^2 Y$ is a htpy equiv. – sim. for $C_n Y \to \Omega^N\Sigma^N Y$.

– says: htpy type of these spaces encoded in E_n / mult of loops.

– $C_2(k)$:
- up to htpy, same as pt configurations;
- as $n \to \infty$, $\mathrm{Emb}(\langle k\rangle, \mathbb{R}^n)$ gets sharper $\geq k$. → contractible models for $E\Sigma_k$.

for any Y, (not nec. connected) take gp. completion

$\Omega B(C_n(Y)) \overset{\text{always}}{\simeq} \Omega^N \Sigma^N Y$. $\left(\operatorname{hocolim}_k F(\mathbb{R}^n, k) \times_{\Sigma_k} X^k\right) \cong = F(\mathbb{R}^n,\infty)_+ X^\infty$!

Cor: $Y = X_+$, X conn, then $\times\, \mathbb{Z} \overset{H_*\text{-ism}}{\simeq} \Omega^N\Sigma^N(X_+)$. (like GMTW models.)

$(\pi_0(l) \cong \mathbb{Z})$

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when $X = *$, get $\Omega^n S^n$ → Σ_n's get homological model, at $\Omega^\infty S^\infty$,

Braid → $\Omega^2 S^2$.

- Compute homology of operad: e_n, $P(n)$:
  - consider $F(\mathbb{C}, n) \xrightarrow{\phi_{i,j}} \mathbb{C}^\times \simeq S^1$ (each i, j)

    $(z_1, \ldots, z_n) \mapsto z_i - z_j$

  Let $\omega_{ij} \in H^1(F(\mathbb{C}, n); \mathbb{Z})$ be the map defined by these maps.

  Arnold: for $1 \le i \ne j \le k$, $H^*(F(\mathbb{C}, n); \mathbb{Z})$ is a graded-comm ring, w/ gens $\omega_{ij} \in H^1$, & relations • $\omega_{ij} = \omega_{ji}$; & • $\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$ - Jacobi-like? ($\ell = 0$?)

  (Birman's book on braids & mapping-class grps:
  - $H^1(\ldots) = \mathrm{Hom}(PB_k, \mathbb{Z})$ - pictures of dual braids).

Consequence: Compute Poincaré series of H^*_k:

Let $b_k = H_*(F(\mathbb{R}^2, k))$ - operad in gr Vect$_k$. (graded alg's)

- $p(t) = \sum_{i \ge 1} \dim H_i(=b_i) t^i = (1+t)(1+2t)\cdots(1+(k-1)t)$.

Thm: (F. Cohen) ∃. natural equivalence btw. algs over this operad & braid algebras.

pf: Main pt: ∃ nat. correspondence btw elts of $H_k(F(\mathbb{C}, n))$ & expressions formed from k elts $\{a_1, \ldots, a_k\}$ of a braid alg. s.t. each elt occurs exactly once. (combinatorial flavor).

Ex: $k = 1$: word in 1 var a_1 is $\{x\}$. - deg 0: equiv to $H_0(F(\mathbb{R}^2, 1)) \cong \mathbb{Z}$ - one gen.

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$n=2$: words in 2 vars x, y: Xrank whichever degree class

- deg 0: $x \cdot y$ (up to sign);
- deg 1: $[x, y]$ — that's it. $\to H_0(F(\mathbb{R}^2, 2)) \cong \mathbb{Z}$.
  $H_1(\quad " \quad) \cong \mathbb{Z}$. $\to$ circle

$n=3$: words in x, y, z:

deg 0: $x \cdot y z$

1: $[x,y]z$; $[y,z]x$, $[x,z]y$. $H_*(F(\mathbb{R}^2, 3)) = \mathbb{Z} \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^2$

2: $[[x,y],z]$; $[[y,z],x]$ — that's all by Jacobi.

— generally, this is the pf — constructs of these classes. in $H^{*}(-)$.

BV-alg characterization (Getzler) — equally elementary:

— makes an F-Cohen then to start.

— $(\widetilde{BV})(n)$ — should be vector space spanned by words in free BV alg generated by n-variables $\{x_1, \ldots, x_n\}$ — formally.

claim: $(\widetilde{BV})(n) \cong H_*(P(n))$. ← BV-alg. operations.

pf: $bv(n)$ spanned by: $P_1 \times \{\Delta^{\epsilon_1} x_1, \ldots, \Delta^{\epsilon_n} x_n\}$, $\epsilon_i = 0, 1$. ($P_1 \in b_{n_1}$, each)

— why? b/c of BV relations: if apply Δ to a product $\to$

- $\Delta(a \cdot b) \Rightarrow$ bracket $+ \pm a \cdot \Delta(b) + \pm \Delta(a) b$, etc
- $\Delta([a,b]) = [\Delta a, b] \pm [a, \Delta b]$. by defn of bracket ($\&\ \Delta^2 = 0$).

- each $\Delta^{\epsilon_i} x_i$ represents $H_*(S^1)$;

so $bv(n) = H_* F(\mathbb{R}^2, n) \otimes H_*(S^1)^{\otimes n} = H_*(P(n))$. ∎.

Utility of these operads: if M any (closed) (oriented) mfld, $F(M, n) \cong$

$\mathrm{Diff}^+(M) / \mathrm{Diff}^+(M, \{x_1, \ldots, x_n\})$: $\mathrm{Diff}^+(M)$ acts transitively on the config space;
fixed pts.

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& the action has s[illegible]: $\Rightarrow F(M,k) =$ single orbit of action $\cong G/\text{stabilizer}$.

if M has bdry, $F(D^2,k) \cong \mathrm{Diff}^+(D^2,\partial) / \mathrm{Diff}^+(D^2, \{x_i\} \cup \partial)$. By same arg.

if $M =$ disc $: D^2$

– Smale: $\mathrm{Diff}^+(D^2,\partial) \simeq *$. So $F(D^2,k) \simeq B\mathrm{Diff}^+(D^2,\{x_i\}\cup\partial)$

knew already that $F(D^2,k)$ a $K(\pi,1)$: $(K(PB_k,1))$

so $PB_k \cong \pi_0 \mathrm{Diff}(D^2,k,\partial)$, & $\cong \mathrm{Diff}(D^2,k,\partial)$. (be a $K(\pi,1)$)

so components of $\mathrm{Diff}^+(D^2,k,\partial)$ are $\simeq *$.

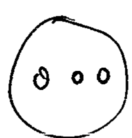
sim arg: $F(D^2,k)/\Sigma_k = K(B_k,1)$; $\cong B\mathrm{Diff}^+(D^2, \{k,\text{pts}\}, \partial)$ (setwise)

BV alg: k-to-1 pair-of-pants $\to$

claim: $P(k)$ ($\cong F(D^2(k))$) is a model for $B\mathrm{Diff}^+(P_k,\partial)$; is a $K(\pi,1)$;

where $\pi_k = PB_k$.

Then a TCFT: assigns to S^1 a chain cx $A(S^1)$; $H_*(A(S^1))$ is an alg over $H_*(B\mathrm{Diff}^+(P_k,\partial))$ (by restriction) $= H_*(P(k)) \Leftrightarrow$ BV-alg.

redraw pairs  P_k;

consider config space $\mathcal{F}_k = \{(x_1,v_1),\dots,(x_k,v_k)$: $(x_1,\dots,x_k) \in F(D^2,k)$; each $v_i \in T_{x_i}M \setminus 0\}$.

– same arg: $\mathrm{Diff}^+(D^2,\partial)$ acts transitively on $\mathcal{F}(k)$; so $\mathcal{F}_k \simeq \mathrm{Diff}^+(D^2,\partial)/\mathrm{Diff}^+(D^2;\partial,\text{fixed}\{(x_i,v_i)\})$ $\simeq F(\mathbb{R}^2,k) \times (S^1)^k \simeq P(k)$.

– action on v_i is by d (differential).

$= B\mathrm{Diff}^+(D^2,\partial,\{x_i,v_i\})$

$\to$ a. [illegible] diffeo fixing pt + direction:

isotopic to id + twist 

– $\simeq B\mathrm{Diff}$ taking diffs fix little disks $= P(k)$. [no #71 – earlier]

$= \mathrm{Diff}^+(P_k,\partial)$.

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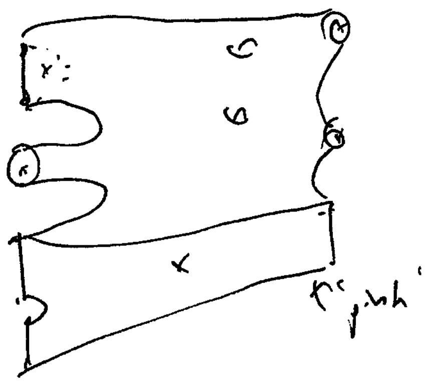
Change of subjects: open-closed TCFTs next up.

– classification – lower-tech than Lurie's lectures

– paper: K. Costello, "TCFTs, and Calabi-Yau categories"

1st: open-closed theory: consider cobordisms btw. cpt mflds w/ bdry

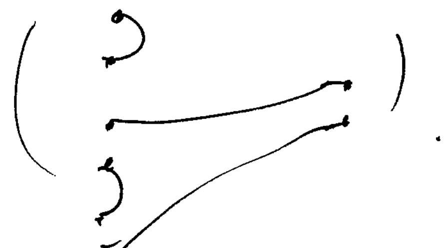
(1st: 1-mflds w/ bdry) :



– bdry 'circles' between 1-mflds

$\partial\Sigma = \partial_{in}\Sigma \cup \partial_{out}\Sigma \cup \partial_{free}\Sigma$

1-mflds w/ bdy; cob bdry btw $\partial(\partial_{in}\Sigma)$, $\partial(\partial_{out}\Sigma)$.



• consider M.. set intervals w/ bdry conditions → Gromov-Witten thy.

– bdry conditions: $)c_1 \quad c_4$, $)c_2 \quad c_3$

~SFT

← Lagrangian submflds. SFT

• ... (STI)

• string thy: cl. strings (~bundles). – D-branes

Greg Moore – Segal paper o-c

– 'easiest' topological QFT.

– arXiv:

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