M283
2/14/08
W #12

Goal: classify O-C TCFTs in 2 dim's. (Costello/H-L)

- ~~using~~

1st: classify 2-D O-C TQFTs (Moore-Segal):

Q's: ① given a comm. FA $\mathcal{E}$, (i.e., 2D-closed TQFT) and a set Λ (D-branes), what are the possible choices of V_{λ_1,λ_2}, λ_i FA s.t. they form an open-closed field theory?

A: twisted (cat'l) K-theory of $\mathcal{E}$ (avoid this for now - running out of time)

② Given an open TQFT, (i.e. open & cobordism cat = Cob_2 w/ objects (intervals w/ labels) w/ labels. - no closed components.
• have FA $\mathcal{O}$ - $\mathcal{E}$ ~~Ch~~ $Ch(\mathcal{O})$

- What are O/C compatible TQFTs? - extend to Fro cob$_2$.

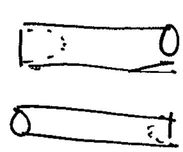
Important Case: $\Lambda = \{\lambda\}$ - 1 D-brane. → one V.s. $V = V_\lambda^\lambda$

- V a (not nec. comm) FA/k, $\mathcal{E}$ = comm FA.

+ other data: , ∘, ∩ give FA struct to V, ; standard non-degen
$V\otimes V \to k$ → $V \xrightarrow{\cong} V^*$ + coproduct

• comm FA $\mathcal{E}$ → (closed) ops. ~~for~~ • p, discs, etc: - μ_c, θ_c, 1_c, $\langle , \rangle_c$
$\nu: \mathcal{E} \to \mathcal{E}^\vee$; $\psi: \mathcal{E} \to \mathcal{E}\otimes\mathcal{E}$.

• relations: (cylinders:)



~~map~~ $V \xrightarrow{\iota^*} \mathcal{E}$, & $\mathcal{E} \xrightarrow{\iota_*} V$ (in $Z(V)$).

Thm. Morse theory: every open-closed cob. can be decomposed into these pieces. - so 2 FAs + maps give all ops;
- check relations corr. to independence of decomposition of cobordism.

relations: ok a map of alg's :

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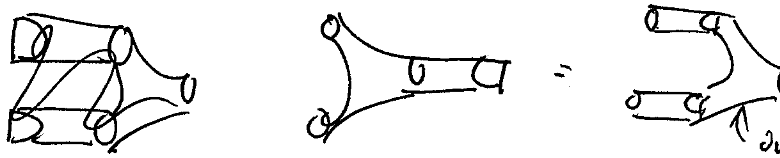
$i_K(\alpha)\cdot i_K\beta$: (1)



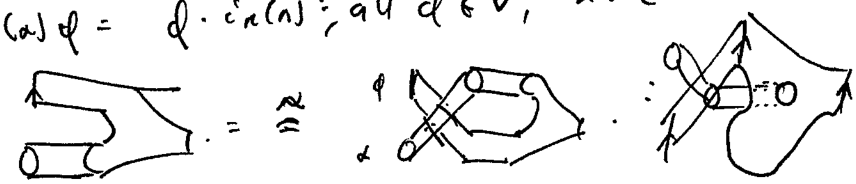
$= i_K(\alpha\cdot\beta)$ ↑ one S^1 bdry:

diml isomorphic rel ∂ to

: both have one bdry component attached.

(2). image of $i_K \subset Z(V)$: picture:

$i_K(\alpha)\phi = \phi\cdot i_K(\alpha)$; all $\phi\in V$, $\alpha\in\mathcal{E}$.

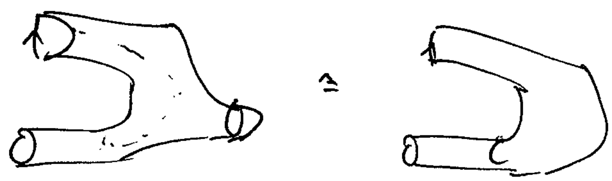


$Z(V)$

center same minimal comm. alg in V.

• other relations: (3) i^K is adjoint to i_K:

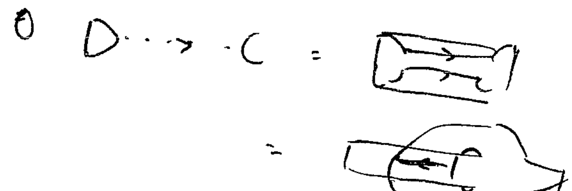
$$\langle i^K\phi,\alpha\rangle_{\mathcal{E}} = \langle \phi, i_K\alpha\rangle_V :$$



~ ∂V.

(4): Cardy condition:

Thm (Moore-Segal) an O-C TQFT w/ 1 ... is the same as:



(1) $\mathcal{E}$ comm FA, V a not nec comm FA

(2) 2 hom adjoint hom's ... $i_K: \mathcal{E}\to V$, $i^K: V\to\mathcal{E}$.

(3): if $\{\psi_1,\dots,\psi_n\}$ basis for V, $\{\psi^1,\dots,\psi^n\}$ dual basis wrt $\langle,\rangle_0$,

then $i_K i^K(\phi) = \sum_{i=1}^{n} \psi^i\phi\psi_i$:

↑ internal expression in V:

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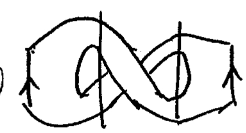
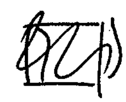
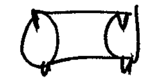
show this, defn:

let $T: V \to V$ = "double twist" as: φ  ($\cong$ )

$\Delta \circ T \circ \mu$.

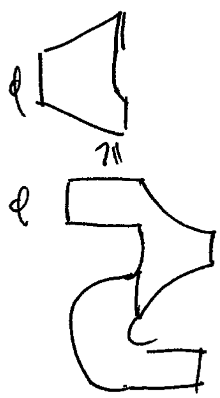
cylinder.



- claim: if $\varphi(1_\sigma)$ known, know $\varphi(\varphi)$ as well:

~~$= \varphi(1)\cdot\varphi(\varphi)$~~ ↑ φ as A-module map

Δ

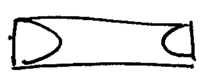


$$\Delta(\varphi) = \varphi\cdot\Delta(1) = \sum \varphi\, \psi_i \otimes \varphi^i$$

$= \mu(\varphi$

apply T: $\sum_j \psi^j \otimes \varphi\psi_j$

$\mapsto \sum_j \psi^j \varphi \psi_j$.

reduced double twist: claim same as  :

- exercise: 

Costello thm: some generalization to TCFT w/ Cat's of D-branes (Λ).

- rest of term: go through Costello's paper: (derived functors language).

- categorical algebra.

overview:

· within O/C TCFT, do have FA's (NC FA) - start w/ NC FA V

- classify O/C TCFTs including V as its NCFA?

(ref: K. Costello, arXiv, TCFTs and Calabi-Yau Categories).

(generalizes FA).

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recall category $\mathcal{M}_\Lambda$ w/ objects = 1-mfds, bdry is labeled by Λ (components)

• sets of morphisms are moduli spaces of RS's giving open-closed cobordisms b/w (with)

$\mathcal{OC}_\Lambda = C_*(\mathcal{M}_\Lambda)$ – same objects, w/ hom cx's = chains of moduli spaces.

– cat • enriched over ~~...~~ chain cxs /k (Ch(k)). ← field

$\mathcal{O}_\Lambda \subset \mathcal{OC}_\Lambda$ is full subcat on objects w/o closed components.

$\mathcal{C}_\Lambda \subset \mathcal{OC}_\Lambda$ not full subcat. – no open components.
– ~~...~~ – no free bdry – assign $\mathcal{C}_k M$ category.

• An open-closed TCFT is a monoidal functor from $(\mathcal{OC}_\Lambda, \mathcal{O}_\Lambda, \mathcal{C})$ to Comp(k).

Goal: compute $\mathrm{Fun}^\otimes(\mathcal{OC}_\Lambda, Ch(k))$. (or iso classes thereof?)

– strategy: compute extensions

$$\begin{tikzcd} \mathcal{O}_\Lambda \arrow[dr, "\otimes V"] \arrow[d, hook, "i"'] & \\ \mathcal{OC}_\Lambda \arrow[r, dashed, "F"'] & Ch(k) \end{tikzcd}$$
$\otimes$.

$i: \mathcal{O}_\Lambda \to \mathcal{OC}_\Lambda$ determines restriction

$i^*: \mathrm{Fun}^\otimes(\mathcal{OC}_\Lambda, Ch(k)) \xrightarrow{i^*} \mathrm{Fun}^\otimes(\mathcal{O}_\Lambda, Ch(k))$ – want adjoint maps.

(really, extension map ... – physics: integrate).

– want $i_*: \mathrm{Fun}^\otimes(\mathcal{O}_\Lambda, Ch(k)) \xrightarrow[\text{("induction")}]{\text{sections}} \mathrm{Fun}^\otimes(\mathcal{OC}_\Lambda, Ch(k))$.

(need HA for this)

Motivation: A – algebra;
Consider cat C_A – 1 object, $*$, morphisms = A (enriched/Ch(k) (cat).)

• functor $F: \mathcal{C}_A \to Ch(k)$ is a (DG) A-module M for DGAs

(conv cats/enriched cat'l)
$F(*) = M$; $A \to \mathrm{End}(M, M)$
$\Leftrightarrow A \otimes M \to M$

- subalgebra $A_1 \xrightarrow{i} A_2$ — have $i^*: F(\mathcal{C}_{A_2}, Ch(k)) \to F(\mathcal{C}_{A_1}, Ch(k))$
$\quad C_{A_1} \hookrightarrow C_{A_2}$ + ~~~~ induction map: (extension).
$i_*: F(\mathcal{C}_{A_1}, Ch(k)) \to F(\mathcal{C}_{A_2}, Ch(k))$

Send M to $A_2 \otimes_{A_1} M$ (extend scalars).
- functor. — dg analogue of PF.

- notation: $i_* f := \mathcal{O}\mathcal{C}_\Lambda \otimes_{\mathcal{O}_\Lambda} f$ (need details here).

But $A_2 \otimes_{\mathcal{O}_1} -$ not exact — (has derived fns.

- replace M w/ free A_1-modules — to preserve structure.
→ (cofibrant replacement notion).

- $\mathbb{L} i_*$ derived version; $\mathbb{L} i_* f = \mathcal{O}\mathcal{C}_\Lambda \otimes^{L}_{\mathcal{O}_\Lambda} f$ making this exact.

- ~~~~ topological analogy to G-spaces. - Replace $M \times_G N$
w/ $M \times EG \times_G N$. (cofibrant spaces).

- model cats on $Gr\,Top^G$. (the version)
struct

Thm (Costello): 1) category of open TCFTs ~~~~ is h.e. to
category of "Calabi-Yau categories". (cat. analogue of FAs)

2) $\mathbb{L} i_* f$ ~~~~ $: \mathcal{O}\mathcal{C}_\Lambda \to Ch(k)$. is ~~~~ "$H_*$-split" - is a strict
monoidal functor ~~~~ on $H_*(-)$.

3) $H_*((\mathbb{L} i_* f)(S^1)) = HH_*(f)$ (Reads CH by cats of structure)
$\quad\uparrow$ in cat. sense.

4) $\mathbb{L} i_*$ is universal. (in some sense).

Apps to Fukaya cat in (open cat) G-W theory.

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