M283
2/28/08
cw HTTQ

Recall technical construction from last time:

- machinery for dg sm cats / k, k a field, char $k=0$.

for A a dg sm, a left A-mod is a monoidal functor $F: A \to$ ChCx (Comp(k))

i.e. NTs $F(A \otimes_A B) \leftarrow F(a) \otimes_{ch} F(b)$ - not isoms.

∘ notion of quasi isoms ⇒ $F: C \to D$ taking qisms to q-isms called exact.

(usual notion in chain CAT)

• an A-mod is flat if $- \otimes_A M : A^{op}$-mod $\to$ Comp(k) is exact

(reduces to usual notion for modules / chcx - replace M w/ acyclic projective resol.)

→ derived tensor product.

Thm: Suppose A a DGSM w/ unit I for $\otimes$, and the monoid of objects (taken?) is a free monoid. (think cob. cat - free on I, S^1 generators).

Then ∃ a functor $F: A$-mod $\to A$-flat s.t.

$F \circ i \simeq id$, $i \circ F \simeq id$ ($\simeq$: quasi isom). (in - NT s/c map of functors).

Pf. recall parts.

- consider $\mathrm{Ob}\,A$ as an SM-subcat of A - only morphisms are id, permutations. - free on generators by hypothesis on A + char $k=0$.

Lemma: $- \otimes_{\mathrm{Ob}A} - :$ ObA-mod × mod-ObA → Comp(k)

is exact in each variable.

• objects are free; cat is a groupoid; so isoms of a given object is a subgp of Σ_n. (preserving order of factors)

→ hence is a rep'n of Σ_n (or subgp?).

~~if W is a~~ consider $V \to V \otimes_{k[\Sigma_n]} W : \mathrm{mod}\text{-}k[\Sigma_n] \to \mathrm{Vect}(k)$ / $\mathrm{Comp}(k)$.

Then this is exact if $\mathrm{char}\, k = 0$: replace W by a free resolution $F_\bullet \to W$ (of $k\Sigma_n$-modules)

- and get q.i. of $k\Sigma_n$-modules b/c $H_*(G; k) = 0$ as (?)

Let $C = \mathrm{Fun}(\mathrm{ob}A, \mathrm{Comp}(k))$. (category) $\exists$ rel. forg. map $G : \mathrm{ob}A\text{-mod} \to C$ ($\mathrm{Fun}^{\otimes} \to \mathrm{Fun}$).

Lemma: G has a left adjoint $F : C \to \mathrm{ob}A\text{-mod}$ which is exact.

- explicit descr.: define $F(V)(a) = F(V)(a_1 \otimes \cdots \otimes a_n)$ by:

for each $I \subset [n]$ define $a_I = \bigotimes_{i \in I} a_i$; , a different I.

$$F(V)(a) = \bigoplus_{k \geq 0} \Big(\bigoplus_{[n] = \{I_1, \dots, I_k\}} V(a_{I_1}) \otimes \cdots \otimes V(a_{I_k}) \Big)_{\Sigma_k} - \otimes_{k\Sigma_k} k$$

parts - ↑ coinvariants of Σ_k action

Let $T = G \circ F : C \to C$. - observe T is a triple: (monoid in cat of functors)

(recall a triple mean SMC is a functor $T : C \to C$ w/ NTs $T \circ T \xrightarrow{\mu} T$, $I \xrightarrow{\eta} T$ w/ monoid diagrams.

- can construct whenever have such adjoint fun.

- an algebra over a triple is a functor $M : C \to C$ w/ NTs $TM \xrightarrow{a} M$, st. (assoc. functor)

Ex: take tensor algebra on cat of vector spaces $V \mapsto \bigoplus_{n \geq 0} V^{\otimes n}$.

(monoid, actions in cat of functors).

alg. over this T = same as $\mathrm{ob}A$-mods. (exercise).

Lemma: $V \mapsto A \otimes_{\mathrm{ob}A} TV$ left adj. to $G : A\text{-mod} \to C$. (free A-mod)

so $\hom_A(A \otimes TV, B) \cong \hom_C(V, GB)$.

Let T_A = can. triple for this adjunction (on $C \to C$).

– note algebras over $\underline{T_A}$ are isom to A-mods.

Lemma: T_A is exact. ($C \to C$)

Lemma: Assignment $A \to T_A$ is exact. (q isom Sta & D(A) say give q isoms btw triples $T_A \to T_B$).

• if $A \to B$ a quasiisom (cat of DGAs), then $A \otimes_{0 \cdot A} M \to B \otimes_{0 \cdot A} M$ is ~~exact~~ a quasiisom, $\underline{\text{any}}$ M.

• given a ~~right~~ left A-module M, define $Bar_A(M)$ $\underline{B}$-flat A-mod, by:

$$Bar_A M = T_A\Big(\bigoplus_{i \geq 0} T_A^{\,i} M\ [-i]\Big)$$ ($[-i]$: shift) · claim: $Bar_A M$ is $\underline{\text{flat}}$.

– pf: define filtration $F^k Bar_A M = T_A\Big(\bigoplus_{i \geq 0}^{k} T_A^{\,i} M'\Big)$ (shifts)

– assoc. graded object is $A \otimes_{0 \cdot A} T\Big(\bigoplus_{i \geq 0} T^i M\Big)$

if N a rt. A-mod, $N \otimes_A Bar_A M$ has can. $\overset{\text{g.p.}}{\text{filt.}}$;

if $N \to N'$ a quasiisom. wts. induced maps $N \otimes_A Bar_A M \to N' \otimes_A Bar_A M$ etc

are quasiisoms too. – suffices to examine assoc. graded object.

– clear by prev result about T (vs. T_A). $\square$

Derived tensor product: $M \otimes_A^L N = M \otimes_A Bar_A N$; So $- \otimes_A Bar_A N$ ~~flat~~ exact,

takes $\underline{\text{q.i. to q.i.}}$

(like model-cat construction in O-spaces –)

$X \times_G^L Y := X \times_G (EG \times Y)$ – $- \times_G (EG \times Y)$ preserves (weak) eq. types.

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Pushforward: if $F: A \to B$ a functor, (DGSM) let N be a left A-module,

(pushforward) $(Lf_*) N$ a left B-mod (Mukhanov $f_! N$).

$= B \otimes_A^L N$. — treat B as a B-A-bimod.

- Back to TCFTs: almost.

Defn: a quasi (i.e. homotopy equivalence) btw 2 DGcats is $f: A \to B$ is a functor s.t. $H_* f$ is "fully faithful": induces isoms

$$\mathrm{Hom}_A(a_1, a_2) \longrightarrow \mathrm{Hom}_B(fa_1, fa_2) \text{ induces an isom in } H_*,$$

for each pair (a_1, a_2). & f is a 'bijection' on objects (isom) — so small cats?)

— all notions of cats cats 9-isos (as always are apply. always C_*.

— Thm: (htpy invariance) if $f: A \to B$ is a quasi-isom of DGcats, then Lf_*, f^* are inverse quasi-equivalences of cats $\underset{A\text{-mod} \leftrightarrow B\text{-mod}}{}$

Now back to TCFTs: - models are $\mathrm{Sk}(M\hat{\mathrm{cob}})$ + extra data.

recall a monoidal functor $F: A \to B$ has $\overset{NTs}{F(a, b)} \leftrightarrow F(a) \otimes F(b)$.

(e.g. EZ, AW in $C_*(\mathrm{top})$). - such a functor is strong (MacLane) or split (Costello) if they are isomorphic; h-split (Costello) if they are q-isomorphic. (watch this for confusing terminology).

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Defn: ① An open-TCFT is a pair (Λ, ϕ), where Λ a set (D-branes), & ϕ is a O_Λ-mod. that is H-split.

2) a morphism of open TCFTs is $(\Lambda, \phi) \to (\Lambda', \phi')$ – set map $\Lambda \xrightarrow{f} \Lambda'$, & morphism $\phi \to f^*\phi'$ of O_Λ-mods. ftn. induced $O_\Lambda \to O_{\Lambda'}$.

– same for open closed

H-split: if ϕ a closed TCFT, can define $H_*\phi$:

– As has n-components: have iso $H_*(\phi(c)) \xrightarrow{\cong} H_*(\phi(S^1))^{\otimes n}$.

so define $(H_*\phi)(c) = H_*(\phi(1))$. – determines H_n into chain level.

map: $H_*(\overbrace{\mathrm{Mor}_{\mathcal{C}}(n,m)}^{\text{mod. space}}) \mapsto \mathrm{Hom}(H_*(\phi)^{\otimes n}, H_*\phi^{\otimes m})$ w/ sign.

(closed cob. cat.)

Thm: • can classify OC TCFTs (via C-Y cats).

• adjoint to O TCFT → OC TCFTs.

. $HH_*(f_! A) = HH_*(A)$.

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