M283
3/4/08
LN #15

· Proving Costello's main Thm: 2-3 lectures ; then examples in rem. lectures.

Thm (Costello): ① the category of open TCFTs is quasi-equiv. to the cat. of unital Calabi-Yau (extended A_∞) categories.

[Calabi-Yau cat: linear cat w/ traces for objects:
· tr_a: $Hom(a,a) \to k$ s.t. pairings $Hom(b,a) \otimes Hom(a,b) \xrightarrow{\mu} Hom(a,a) \to k$ is symmetric, nondegen. ∃ ← "FAs w/ many objects" ← D-branes.

②. Given any open TCFT, (Λ, ϕ), $\phi: \mathcal{O}_\Lambda \to Comp(k)$ monoidal, then $\mathbb{L}i_*(\phi): \mathcal{OC}_\Lambda \to Comp(k)$ is h-split, hence defines an OC-TCFT. - "universal in an approp. sense"; from Left adj to res.

③. Homology of $\mathbb{L}i_*\phi$ is $\widehat{HH}_*$(corr. A CY cat) to (Λ, ϕ).

(recall Moore-Segal result: 1 D-brane; then CY cat is an FA; h-split. the $HH_*(A)$ is $H_*(E(S^1))$).

FA $\xrightarrow{A}$ CY cat → O TCFT → OCTCFT $\xrightarrow{H_*}$ $H_*(S^1) \cong HH_*(A)$.

· $C^*(M)$ is 'favorite' FA (up to htpy) ⟼ $HH^*(C^*M) = H_*(E(S^1))$

→ string top as example.

· want sort model for $\mathcal{O}_\Lambda$ category. - replace $C_*(M(i,j))$ by a given cx. :

· cell decomp of $M(i,j)$. ← combinatorics.

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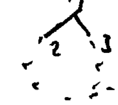
• lots of moduli: – often arise from spaces of ribbon graphs.
for moduli space.

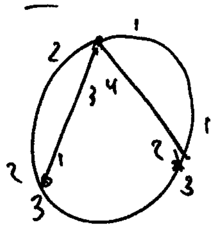
- cell decomp of ~~mod~~ moduli space: fat or ribbon graphs: finite graph, ~~+ cyclic~~

~~orderings of~~ – each vertex is of ≥3 valent; (extra constraint) 

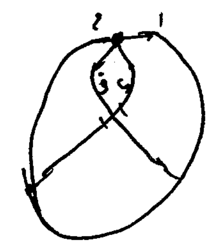
– each vertex has a cyclic ordering of its 1/2-edges. (extra data) 

(local info).

picture: (CCW ordering @ each place)



vs.



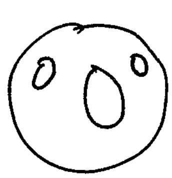
– reorder edges

– picture:

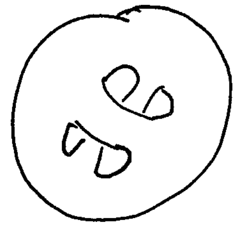
– X →)(

already vertex, only edges.

thicken:



vs.



(genus 0, $|\partial| = 4$)

$|\partial| = 2$, $g = 1$.

– htpy equiv surfaces, not diffeo

• topological space of graphs – with lengths on edges ("space of metric fat graphs").

(of type (g, δ)) (g = genus, $\delta = |\partial|$ mod.)

– can be viewed as $B\mathcal{C}^{\text{fat-graphs}}_{g,\delta}$ (classifying space of cat of (g,δ)-fat graphs)

– obj $\mathcal{C}^{\text{fat}}_{g,\delta}$ = fat graphs

mor – gen by collapses of "forests": disjoint union of trees,

– includes edge collapse. – g, δ unchanged – each cyclic ordering, too – obviously a htpy equiv. (collapse)

– cells in Δ^k's ↔ metrics on edges.

(Véronique Godin's thesis studies these objects)

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Thm (Penner; Thurston, Harer, Kontsevich, Godin) $B\mathcal{C}_{g,\delta}^{top} \simeq \mathcal{M}_{g,\delta} \times \Delta^{\delta-1}$ (stab. thm. — Subst. to later ?'s.)

$\sim \mathcal{F}_{g,\delta}$

δ = # of punctures / marked pts on genus g - R.S.

So $\simeq M_{g,\delta}$; $B\mathcal{F}_{g,\delta}$ has cell decomp (

- strata of metric fat graphs - via edge / vertex ?. (Harer / Ditto;)

→ fun cells ?.

Costello: similar decomp - w/ categorical ? of M. (Kuiz, etc.)

(note on Harer stability - H_* - indep of g, in a range).

- Same uses similar spaces of graphs to compute $H_*(B\mathrm{Aut}(F_n))$)

Defn (Costello): let $\alpha \in Ob\,OA$, $\beta \in ob\,OC_n$. to describe a cell decomp of $\tilde{M}(\alpha,\beta)$ (span) ($\mathcal{E}(\alpha,\beta)$.)

①. def: $\overset{\sim}{\eta}(\alpha,\beta)$ = mod. sp. of RS's Σ w/ arbitr. closed bdys, labeled $0, \ldots, c(\beta)-1$ (# of closed pieces); each closed o.g. bdy has exactly 1 marked pt. (replaces parametrization of bdy) (so also labeled) - up to htpy - $\mathrm{Diff}^+(S^1) \simeq S^1$).

$\exists$ marked pts labeled by $O(A)$, $O(\beta)$, distrib. among remaining bdy comps.

- Σ may have nodes along bdy ; no nodes in interior, no mkd pts in interior.
- mkd pts cannot collide w/ nodal pts.
- no nodes on closed bdys.

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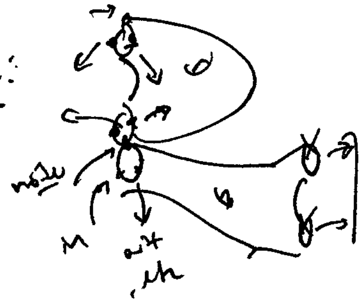
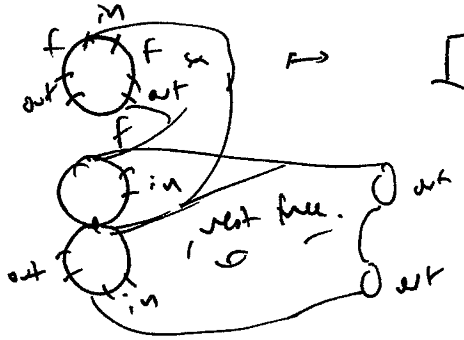
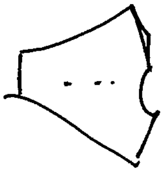
Example: 

disk w/ segs

pts here: denote internal bdys:

redraw:  ⟼ 

- except reduce to pts b/c (?) don't matter.

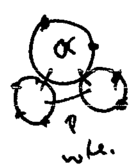
($D; H^+(I) \simeq *$)

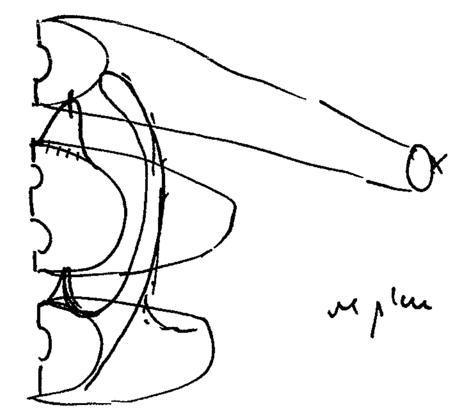
nodes: degenerated genus structure:

simpler subspace: Let $\mathcal{G}(\alpha,\beta) \subset \overline{\mathcal{N}}(\alpha,\beta)$ consist of various $\Sigma \in \overline{\mathcal{N}}(\alpha,\beta)$, such that whose "unit components" is either a disc, or an annulus.

(components w/ output)

require otherwise: that $\underline{1}$ side of each annulus is outgoing (hence, smooth)

picture:  ⟷ cob: 7 o I's to 1 S^1.



replace nodes $X \sim$  - handle

prop: $\mathcal{G}(\alpha,\beta) \hookrightarrow \overline{\mathcal{N}}(\alpha,\beta)$ is a weak h.e. (of "orbispaces") (result of ... [cos. p...])

(which gives $\mathbb{Q}$-h.e. of (coarse) mod spaces)

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Pf: (induction on $c(\beta)$ — # of closed, e.g. Adjoin S^1's)

Assume: true for $c(\beta) = 0$. (do later — Costello paper).

- Let i be $0 \le i \le c(\beta)-1$; Let $\mathcal{D}'(\alpha,\beta)$ be the same space as $\mathcal{D}(\alpha,\beta)$, but w/ no marking pt on ith closed o.s. S^1.

- define $\bar N'(\alpha,\beta)$ similarly. : consider:

$$\begin{tikzcd} G(\alpha,\beta) \arrow[r, hook] \arrow[d] & \bar N(\alpha,\beta) \arrow[d] \\ G'(\alpha,\beta) \arrow[r, hook] & \bar N'(\alpha,\beta) \end{tikzcd}$$

- square is a pullback; prove the map $\to$ a h.e.

let $\beta' = \beta \setminus$ ith S^1; the $\exists$ map $\bar N'(\alpha,\beta) \xrightarrow{g} N(\alpha,\beta')$

by: gluing a disc $D =$ ◯ $=$ (filling S^1.

- also view as filling in disc to marked pt. — ⇒

(Lifting?) fiber (over given diffeo type) is Σ (map to diffeo class) —

- pt can be anywhere on surface.

hence, true for $G'(\alpha,\beta) \longrightarrow G(\alpha,\beta')$ map.

& $G(\alpha,\beta') \longrightarrow \bar N'(\alpha,\beta)$ a h.e (weak) by induction.

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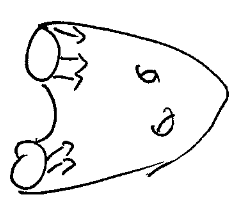
if no outgoing closed bdy: ($c(\beta)=0$). — generic case: $g \ge 2$;

⇒ canonical hyp metric on Σ. ⇒ vector field from metric:



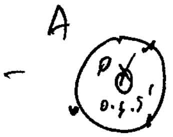
push in along bdy — follow flow from hyp. metric; (1st homology will be weird);

ok in G (??). — need hyp. geom of surfaces, & low genus cases.

→ CW struct by orbicells on $G(\alpha,\beta)$.

(cells $= D^n/G \leftarrow$ finite gp) – fin. gp action – isometric, incl. 0)

Let $\Sigma \in G(\alpha,\beta)$. A be an ~~[illegible]~~ annulus in Σ. (imag compact)



– cut A: – ~~unique~~ biholomorphism b/w A & $S^1 \times I$, s.t. $p \mapsto (1,0)$

→ take min. sm at $1 \times I$ in A:

– simp'l nodal curve.

decompose Σ into cells: 0-skel is nodes, mkd pts, and cut pts on bdries; 1-skel is cuts, $+ \partial\Sigma$; 2-skel is Σ. 2cells are open, width s/c Σ is.

– stratify $G(\alpha,\beta)$: Σ_1, Σ_2 are in same stratum if the corr. marked, w/ 2-cell cxs are isomorphic.

(claim: stratn of $G(\alpha,\beta)$ is an orbi-cell decomp: – so strata are $\cong D^n/G$

• moreover: comp maps are cellular.

– pictures + cases.

New cat: $\mathcal{D}(\alpha,\beta) = C_*^{cell}(G(\alpha,\beta)) \otimes k$. – defines DGSM, qI to $\mathcal{O}_\Lambda$.

(or lattice Λ's; could have added.

↑ strong relation: 2 functors, and qI to id (NT's present).

Goal: study $\mathcal{O}_\Lambda$ via $\otimes$-functors from $\mathcal{D}$ to Cpxs – smaller alg'ic cat.

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