M283
3/6/08
LN#16

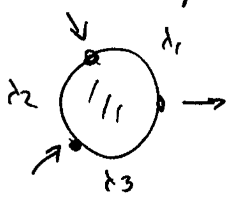
- Moving toward conclusion of Costello paper - pf of main thms

- studying open TCFTs $\leftrightarrow$ CY A_∞-cat structures;

① find cat quasi-isom to $\mathcal{O}_\Lambda$, w/ (nice) cellular structures on $\mathrm{Mor}(\alpha,\beta)$, s.t. composition is also a cellular map.

recall cat $\overline{\mathcal{N}}$: same objects; morphisms $\overline{\mathcal{N}}(\alpha,\beta)$ are surfaces w/ certain marked pts. + nodes allowed

Eg:



represents $2I \longmapsto I$ cobordism;

- but w/o interval parametrizations.

subspace $\mathcal{D}(\alpha,\beta) \hookrightarrow \overline{\mathcal{N}}(\alpha,\beta)$. - subspace of surfaces where each nodal component is either a disc or an annulus;
- annulus has 1 (≥ 1?) outgoing bdy.

- really are ribbon graphs: (correspondence?)

- Main results: · $\mathcal{D}(\alpha,\beta) \hookrightarrow \overline{\mathcal{N}}(\alpha,\beta)$ a htpy equiv;
· natural (cubical) cell decomp of each $\mathcal{D}(\alpha,\beta)$, compat w/ composition.

- 3 types of basic surface cells in $\mathcal{D}(\alpha,\beta)$: - all others are gluing composites of these;

①. discs w/ pts / labels / arrows.

②. annuli: w/ interior circle an o.g. smooth S^1;
- produce outline w/o marked pt on o.g. S^1 - avoids marked pt.

③. - same, but outline hits a marked pt.

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Define an DGSM cat $\mathcal{D}$ - same objects as $\mathcal{O}_\Lambda / \mathcal{OC}_\Lambda$, but

$\mathrm{Mor}(\alpha,\beta)$ are $C_*^{cell}(\mathcal{M}(\alpha,\beta); k)$

- $\mathcal{D}_\Lambda$ is quasi-isom to $\mathcal{OC}_\Lambda$ as DGSMs - smaller model for cat.

(würde: simpler ribbon-graph models for these spaces → better cell decomps).

$\mathcal{D}_{open}$: restrict to full subcat on objects of $\mathcal{O}_\Lambda$.

Now: determine data for $\mathcal{O}_\Lambda^{open} \to \mathrm{Comp}(k)$ to be a TCFT;

- suffices to get for $\mathcal{D}_{open} \to \mathrm{Comp}(k)$.

Steps: (drop Λ notation) - Let $\mathcal{C} \subset \mathcal{D}_{open}$ - subcat w/ same objects,

reduced morphism surfaces: not allowed to have components which are discs w/ ≤ 1 open marked pt.

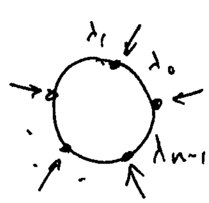
O is D cob; - remove these for now; allow $\emptyset$ O $\emptyset$.

Then $\mathcal{C}$ freely gen by as an (dg) SMcat over the objects of $\mathcal{D}_{open}$

by: $D(\lambda_0, \dots, \lambda_{n-1})$, $n \geq 3$: p's, discs w/

2n bdry mkd pts, subject to the relation $D(\lambda_0, \dots, \lambda_{n-1})$ is

cyclically symmetric;

$D(\lambda_0, \dots, \lambda_{n-1})$ is the disc: 

- cyclic labelling;
- pts at S_n incoming
- all incoming

- takes nI to ϕ :

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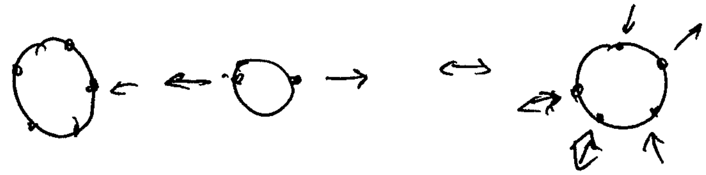
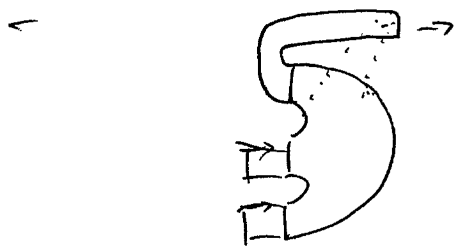
gluing: allow  aswell:

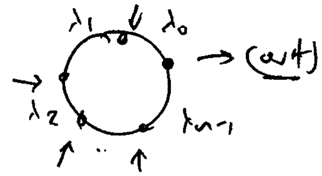


— switch which pt ~~[illegible]~~ to outgoing.



etc:

• 1 more intermediate category: Let $\mathcal{D}^+_{open} \subset \mathcal{D}_{open}$ have same objects, restricted morphisms: morphisms given by $\sqcup$ of discs, each w/ conn. component having exactly one outgoing bdry.

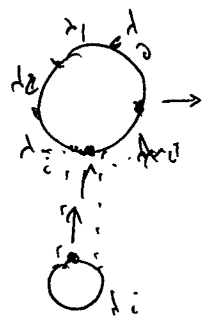
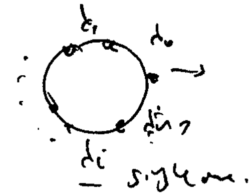
Ex: $D^+(\lambda_0, \dots, \lambda_{n-1})$:  (out) $\in$ Hom$(\{\lambda_0, \dots, \lambda_{n-1}\}, \{\lambda_0\})$

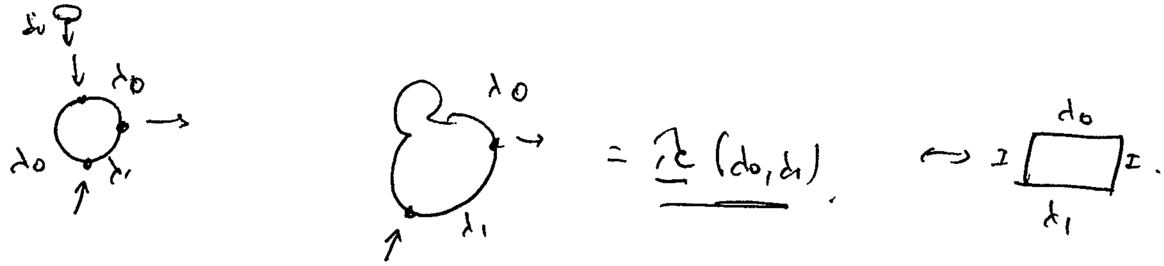
Lemma: (1): $\mathcal{D}^+_{open}$ is freely gen. as a SMC over ob $\mathcal{D}_{open}$, by the $D^+(\lambda_0, \dots, \lambda_{n-1})$, mod: relations

$D^+(\lambda_0, \dots, \lambda_i, \lambda_i, \dots, \lambda_{n-1}) \circ D^+(\lambda_i) = D^+(\lambda_0, \dots, \lambda_{n-1})$, $n \geq 4$

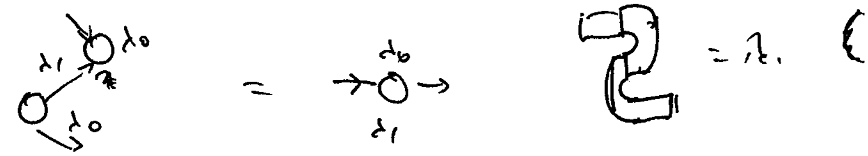
$n = 3$:

$D^+(\lambda_0, \lambda_0, \lambda_1) \circ D^+(\lambda_0) = id$;

(1).  glue $\to$ $\simeq$  λ_i [illegible].



Thm: $\mathcal{D}_{open}$ is freely gen by (OA ...) over Ob $\mathcal{D}_{open}$, by $\mathcal{D}^+_{open}$ (discs) + discs w/ 2 incoming or 2 outgoing bdys., not relations:

① 

② $\mathcal{D}(\lambda_0,\dots,\lambda_{n+1}) = \mathcal{D}^+(\lambda_0,\dots,\lambda_n)\circ \mathcal{D}^{2in}(\lambda_0,\lambda_{n-1})$
(swap out twin).

is cyclically symmetric.

details: check on cells in decomp - cont. realizing set this structure (up to ...)

Now: monoidal functors from these cats.

(small)
- A_∞-cats: (notion of A_∞-alg: Stasheff-(mid 60s) - H-spaces, etc.)
  - cats: 90s - Fukaya)
- A_∞-cat $\mathcal{D}$ (enriched over Ch(k)) has a set of objects Ob $\mathcal{D}$; for each pair A, B, have chain cx $Hom(A,B)$, together w/ composition maps: given $n+1$-tuple $(A_0,\dots,A_n)$, ∃

$$m_n: Hom(A_0,A_1)\otimes Hom(A_1,A_2)\otimes\cdots\otimes Hom(A_{n-1},A_n) \to Hom(A_0,A_n)$$

of degree $n-2$, s.t.:

① m_1 = differential on $Hom(A_0,A_1)$;

② : $\sum_{0\le i\le j\le n-1} \pm m_{n-j+i}(id_i\otimes m_{j-i}\otimes id_{n-j}) = 0$ (next) (assoc up to h.t.py rel'ns)

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• if $\mu_n = 0$, $n \geq 3$, have a category. (strict assoc. of μ_2).

• unitality: assume $\exists$ a unit $1_A \in \mathrm{Hom}(A,A)_0$ for μ_2 (+ effective h.j. higher μ_n's ??).

A Calabi-Yau A_∞-category $\mathcal{C}$ is a unital A_∞-cat w/

maps $\theta_A : \mathrm{Hom}(A,A) \to k$, s.t. pairings

$\langle , \rangle_{A,B} : \mathrm{Hom}(A,B) \otimes \mathrm{Hom}(B,A) \to k$ $(= \theta_A \circ \mu_2)$

are nondegen ; & $\langle , \rangle_{A,B}$ are symmetric;

Symmetry: that $\langle \mu_{n-1}(\alpha_0 \otimes \cdots \otimes \alpha_{n-2}), \alpha_{n-1} \rangle = \pm \langle \mu_{n-1}(\alpha_1 \otimes \cdots \otimes \alpha_{n-1}), \alpha_0 \rangle$

Note: X a CY mfld / $\mathbb{C}$-projective variety w/ $c_1(X) = 0$

- Consider derived Cat of coh. sheaves on X $\Rightarrow$ get CY A_∞-cat.

Serre duality gives the CY A_∞-struct.

• this is CY cat corr to B-model $\simeq$ Ish.

Costello thm: recall a monoidal functor $F: \mathcal{C} \to \mathcal{D}$ has

a N.T : $F(a) \otimes F(b) \to F(a \otimes b)$. - is called split if

these are isos; h-split if quasiisom. ($\mathcal{C}, \mathcal{D}$ enriched / $\mathrm{Comp}(k)$).

Lemma: ① A split monoidal fun $\Phi : \mathcal{D}^+_{\mathrm{open},\Lambda} \to \mathrm{Comp}(k)$ is equiv to

a unital A_∞-cat w/ objects Λ.

Pf: suppose have Φ; for object O, w/ labels $s(i), t(i) \in \Lambda$, $0 \leq i \leq O-1$,

have natl isom $\Phi(O,s,t) \cong \bigotimes_{i=0}^{O-1} \Phi(s(i), t(i))$

for each pair of D-branes, (λ, λ'), define $\mathrm{Hom}(\lambda, \lambda') = \Phi(\{\lambda, \lambda'\})$

- the discs $D^+(\lambda_0, \ldots, \lambda_{n-1})$ give ~~maps~~ ch cx maps

$\mathrm{hom}(\{\lambda_0, \ldots, \lambda_{n-1}\}, \{\lambda_{n-1}, \lambda_0\}) \rightarrow$ defining op onto M_{n-1}, $n \geq 3$ A_∞-ops.

$n = 1, 2$: other struct.

Con 2. Split monoidal fcr $\mathcal{D}_{open, \Lambda} \rightarrow \mathrm{Comp}(k)$ same as a CY-A_∞-cat w/ set of objects Λ.

$\mathcal{D}_{open}$ gen over $\mathcal{D}^+_{open}$ by discs

- same: $\leftrightarrow \langle \, \rangle_{\lambda_0, \lambda_1}$: $\leftrightarrow$ non deg.. ("inverse" - the coeval map).

- two steps = extended CY A_∞-cats. - as a check...

- suppose we have QI $\bigotimes_{i=0}^{n-1} \Phi(s(i), t(i)) \rightarrow \Phi(\mathcal{O}, s, t)$

Defn: h-split sh functor $\mathcal{D}_{open} \rightarrow \mathrm{Comp}(k)$ is called an extended CY A_∞-cat. (split + extended = ordinary).

- no fusion production the more - not EVEs structure.

Cor: Cat of open TCFTs (h-split monoidal fcs $\mathcal{O}_\Lambda \rightarrow \mathrm{Comp}(k)$) is QI to those extended unital CY A_∞-cats.

(By thm def + previous replacement the $\mathcal{O}_\Lambda$ by $\mathcal{D}_{open}$!)

- universal OC TCFT via extension f'dr machinery
  - to Hhx pt next time?

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