M283
3/11/08
LN #17

Today: finish Costello's Thm.

recap: progress on pf:

- if $\alpha, \beta \in \partial \mathcal{S}$, $\mathcal{O}\mathcal{C}_\Lambda$ (1-mfds w/ approp labels), define
  $\overline{\mathcal{N}}(\alpha,\beta)$ cobordisms $\alpha \mapsto \beta$, allowing nodes.

  $\cup$|

  $\mathcal{G}(\alpha,\beta)$ (graphs) : surfaces, built from discs, annuli etc,
  + conditions on annuli bdy.
  + mkd pts + directions.
  $\Leftrightarrow I / S^1$'s

  - orbicell decomp, + weak? $G \to \overline{\mathcal{N}}$ a h.e. (hence, so cell approx to $\overline{\mathcal{N}}(\alpha,\beta)$).
  + comp. cellular.

- 3 types (basic) / cases of cells - ; w/ cut; w/ bits mkd pt.

- $\mathcal{D}(\alpha,\beta) = C_*^{cell}(\mathcal{G}(\alpha,\beta))^{h}_{\Lambda}$ via orbicell decomp.
  - describe cx: $\mathcal{D}(\alpha,\beta)$ - gen by these cell types; (i.e., by these surfaces

bdy maps: increase that nodes.

$$\partial(\bigcirc) = \sum \pm \bigcirc\!\bigcirc + \text{other 'nodal' terms.}$$

$\mathcal{D}$ - not quite a category: - no units for $\beta, \alpha = \emptyset$ - not bdys.

Lemma: $\mathcal{D}_{open,\Lambda} \underset{\uparrow}{\simeq} \mathcal{O}_\Lambda$.
htpy equiv / quasi-isom
→ does not mean $B\mathcal{C} \simeq B\mathcal{D}$. here;
— here: mans. NTS $\mathcal{D} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{O}_\Lambda$,

+ NTS $I \to GF$, $FG \to I$ factor QCs.

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Cor: $\mathrm{Fun}^{\otimes}(\mathcal{D}_{open,\Lambda}, \mathrm{Comp})_{\text{dsplit}} \underset{\text{lin}}{\overset{\simeq \text{ as } \infty}{\rightleftarrows}} \mathrm{Fun}^{\otimes}(\mathcal{O}_\Lambda, \mathrm{Comp})_{\text{hsplit}} \equiv$ open TCFT.

fall's: ~~not~~ don't have HE. ch. spaces as QI a weaker notion

(why? suppose ℓ, ℓ' have HE ch. spaces;

ex: $\underline{G}$ category; vs. $\Omega B\underline{G}$ cat. $\partial\ell$ — gr. cat. 5/e

$\mathrm{Mor}_{\underline{G}}(*,*) \to \mathrm{Mor}_{\Omega B\underline{G}}(*,*)$

But. $\mathrm{Fun}(\underline{G}, \mathrm{Vect}) =$ reps of G;

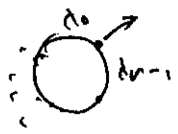
2 HE sp: $\underline{S}^1$, $\mathrm{Diff}^+(S^1)$

easy maps → large — cx maps.

$(\mathrm{Fun}(\underline{S}^1, \mathrm{Vect})$ — rep theory different).

Smaller cat: $\mathcal{D}^+_{open,\Lambda} \subset \mathcal{D}_{open}$ — subcat: same obj;

morphisms are d.g. unions of discs, w/ one outgoing bdy piece



$:= \mathcal{M}_0(\{\lambda_0, \dots \lambda_{n-1}\}, \{\lambda_{n-1}, \lambda_0\})$

Lemma 1. split monoidal fn $\Phi: \mathcal{D}^+_{open,\Lambda} \to \mathrm{Comp}(k)$ same as a unital A_∞ — category, w objects Λ.

$\Phi(\{\lambda_i \lambda_j\}) = \hom(\lambda_i, \lambda_j)$; $\mathrm{Mor}_{\mathcal{E}}(\ldots)$.

Lemma 2: split mon fn $\Phi: \mathcal{D}_{open,\Lambda} \to$ Complexes ~ unital CY-cat. (A_∞ by defns).

— see any $\mathcal{D}^+_{open}$ by →O←, ←O⇒ (units, counit maps).

→ inner product / pairing.

defn: L-split $\Phi : \mathcal{O}\mathcal{C}_A \to \mathrm{Comp}(\mathcal{C})$ is a unital extended CY A_∞-cat.
$\mathcal{D}^+_{\Phi\mu,\Lambda}$

→ Thm: the cat of open TCFTs is "h.e." (A.I.) to cat of unital ext. A CY A_∞-cats.

restriction: If $\mathcal{C}$ an open TCFT ↔ (unital ext CY A_∞-cat)

$(\mathbb{L}i)_\kappa(\mathcal{C})$ an O-C TCFT; and $H_\kappa((\mathbb{L}i)_\kappa(\mathcal{C})(S^1)) = HH_\kappa(\mathcal{C})$ (A_∞ categories)

punchline: universal prop: by adjunction: any OC TCFT Φ w/ restr to $O\Lambda = \mathcal{C}$, then $\exists$ unique map from $\Phi \to (\mathbb{L}i)_\kappa(\mathcal{C})$. (Compare Thm.)

- Changing notes - "CY things"; coh sheaves on CY varieties, etc

recall HH cx: A an assoc alg, M a bimod, f.dim
$CH_\kappa(A, M) = M \otimes A^{\otimes k}$ w/ bdry map given by mult's. (+ diffs for A, M dg mods.)
- normalized version: $\overline{CH}_\kappa(A, M) = C_\kappa(A, M) / D_\kappa(A, M)$ - degen terms - at least 1 $a_i = 1 \in A$.

also makes sense for (small) categories. A a Dla cat; M a bi-fun $A^{op} \otimes A \to \mathrm{Comp}$
$CH_\kappa(A, M) = \bigoplus_{(x_0, \dots, x_n) \in Ob\, A^{n+1}} \mathrm{Hom}(x_0, x_1) \otimes \mathrm{Hom}(x_1, x_2) \otimes \dots \otimes \mathrm{Hom}(x_n, x_0)$ $(x, \underline{A})$
w/ respective bdry maps - well-def b/c assoc. $[1-n]$ shift grading.
- can further reduce, if A unital (exist 1's in $\mathrm{Hom}(x, x)$ in chain cx's). (to normalized case)

. "Cheat!" - Fukaya? embed {Dla cats} ↪ {A_∞-cats} (not comp'h)
is an equivalence. (comparison of cats)

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analogous to: saying we can rigidify homotopy monoids to real monoids up to h.e. ($\Omega X \overset{\simeq}{\leftrightarrow} MX$)

(pf: $\Omega B(X, e, *) \simeq X$. (like $\Omega BG \simeq G$.
↑ monoidal → monoid.

- Fukaya: defines HH of A_∞-cat directly?
- Costello: takes back to dg-cat, avoids higher stuff.

. Let Φ be an unital ext CY A_∞-cat (open TCFT)

. want to study induced closed field thy:

$(\mathbb{L}i)_* \Phi := (\mathcal{O}C_\wedge \otimes^{\mathbb{L}}_{\mathcal{O}_\wedge} \Phi)(1)$ eval on S^1, take HH_*.

$\simeq \quad Bar_{\mathcal{O}_\wedge}(\mathcal{O}C_\wedge)(\theta, 1) \otimes_k \Phi(\theta) \xrightarrow{\otimes^{\mathbb{L}}} \mathcal{O}C_\wedge \otimes_{\mathcal{O}_\wedge} \mathcal{O}J(1)$. Since in proper T.P.

- all $\mathcal{O}C$-$\mathcal{O}\wedge$.

- need (direct sum).

Now: $\mathcal{D}_{open,\wedge} \hookrightarrow \mathcal{O}_\wedge$ qI of cats;

- replace $\mathcal{O}C_\wedge$, $\mathcal{O}_\wedge$ w/ $\mathcal{D}_{open}$. - also $\mathcal{D}_{open}$-free & same

- using only open → open or open to 1.

. $\mathcal{D}_{\mathcal{O}C_\wedge}(\theta, 1) \hookrightarrow \mathcal{O}C_\wedge$ also h.e. (qI 3).

- so consider $\mathcal{D}_\wedge(\bar\theta, 1) \otimes^{\mathbb{L}}_{\mathcal{D}_{open}} \Phi(-)$:
↑ bicoh as $\mathcal{O}C(\theta) \otimes \mathcal{D}_{open}$ - same.

- but two 3.

- $\mathcal{D}_\wedge(-, 1)$ flat as a right $\mathcal{D}_{open}$-$\wedge$ module - no replacement needed.

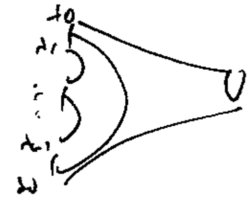
- also assume Φ an honest DG-cat. (Mostly true of CY A_∞-cat)

by replacement.

compute $H_*(D^*_\Lambda(-,1) \otimes_{D_{open,\Lambda}} \mathcal{Q}(H)) \Rightarrow HH_*(\mathcal{Q})$. — replaced + - ..., $\neq 0$.

— look at all levels.

let $A(d_0,\dots,d_n)$, $d_i \in A$ be words ::



— assume these are the only things in $D^+(-,1)$ within D_{open}

$(D^+_\Lambda{}^{(-,1)} \otimes_{D^+_{open}} \mathcal{Q}^{(1)} \cong C_*(B)$.

— general

$$D_\Lambda^+(\{d_0,\dots,d_{n-1}\},1) \otimes_{D^+_{open}} \mathcal{Q}(\{d_0,\dots,d_{n-1}\})$$

$$\cup$$

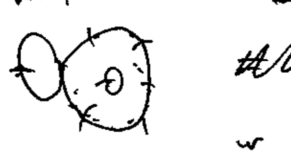
$$\bigoplus_{d_0,\dots,d_{n-1}} A(d_0,\dots,d_{n-1}) \otimes \mathcal{Q}(\{d_0,\dots,d_{n-1}\}) = D^+_\Lambda(-,1) \otimes_{D^+_{open,\Lambda}} \mathcal{Q}(-,1).$$

$$= \bigoplus_{d_i \neq d_0} \mathrm{Hom}_{\mathcal{Q}}(d_0,d_1) \otimes \cdots \mathrm{Hom}_{\mathcal{Q}}(d_{n-1},d_0)$$

is + diff + diff A: — words; — create at most 1 ... — all ...

n=1: — ... diff ...

— bdy:



n=2 — mult

=

q-mult

— using dg cat — vanishes ⇒