Homotopy theoretic reduction of the Kervaire invariant one problem

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Contents

Introduction

This set of notes is a summary of part of my notes on a workshop held in Bonn about the so-called Kervaire invariant one problem.

These notes are certainly sketchy and note complete. They surely need a lot of reworking. They might nevertheless be of some use to someone...

This material is mainly based on the unpublished thesis of J. Jones. It relies also on the original article of W. Browder. Extremely useful is are the Books of W. Browder on Surgery and the Book of

Milnor-Stasheff on Characteristic classes. Finally, the book of Bredon 'topology and geometry' is a solid background reference.

We advertise that there exists a completely independent proof of most of the results here in a short and clear paper of J. Lannes [reference?]. We unfortunately got aware of this reference quit a bit to late for out workshop.

Last bit not least, P. Akhmetiev has recently put online a preprint where he claims to have disproved the existence of Kervaire invariant one elements in high degrees. From this follows that the E_2 cycles h_i^2 in the Adams spectral sequence are for i big enough note infinite cycles.

1 Some prerequisites from Differential topology

The material of this section is taken form [Milnor-Stasheff]

1.1 The Thom isomorphism

Let $\pi : E \longrightarrow M$ be a k-dimensional real vector bundle. We let E_0 be its zero section.

Definition 1.1 *The vector bundle $\pi : E \longrightarrow M$ is* orientable *if for some bundle Atlas, all transition functions take value in orientation preserving linear maps (that is, of positive determinant).*

By definition of a vector bundle, each fiber F_b has a given structure of k-dimensional real vector space hence $H_k(F_b, 0; \mathbf{Z})$ is non canonically isomorphic to $\mathbf{Z}$, the choice of an isomorphism being precisely the choice of an orientation.

We have the following characterization of orientability:

Proposition 1.2 *The vector Bundle $\pi : E \longrightarrow M$ over the connected space M is orientable if and only if $H_k(E, E_0; \mathbf{Z})$ is isomorphic to the integers, and the natural inclusions $(F_b, 0) \longrightarrow (E, E_0)$ induce isomorphisms on the k^{th} integral homology group.*

In the case of a non orientable bundle, we can still work over $\mathbf{F}_2$. We then get:

Proposition 1.3 *For any vector Bundle $\pi : E \longrightarrow M$ over a connected space M, the group $H_k(E, E_0; \mathbf{F}_2)$ is isomorphic to $\mathbf{F}_2$, and the natural inclusions $(F_b, 0) \longrightarrow (E, E_0)$ induce isomorphisms on k^{th} homology with coefficients in $\mathbf{F}_2$.*

The choice of a generator in H^k (coefficients $\mathbf{Z}$ in the orientable case, $\mathbf{F}_2$ otherwise) is called an orientation of the bundle (of course, there is no choice in the non orientable case). Its is determined by the choice of an orientation on a single fiber. Assuming we have made such a choice, then generator is called *the Thom class of the oriented bundle π.*

If M is a nice space (e.g. paracompact), the morphism

$$\phi : H^l M \longrightarrow H^{l+k}(E, E_0)$$

given by the formula

$$\phi(x) = (\pi^* x) \cup u \quad .$$

is an isomorphism called *the Thom isomorphism.* Of course, this is not at all obvious that this map is actually an isomorphism. We will admit it and rely on [Milnor-Stasheff]. Here, the coefficients depend of course of where the orientation class lives (in $\mathbb{Z}$ or in $\mathbf{F}_2$).

1.2 Wu classes

Let M be a closed manifold. We work over the field $\mathbf{F}_2$. By Poincaré Duality, the homomorphism:

$$Sq^k : H^{n-k} M \longrightarrow H^n$$

is of the form:

$$Sq^k x = x \cup v_k$$

for some fixed and natural class v_k. The class

$$v = \sum v_i$$

is by definition the total Wu class of M and we have

$$Sq(v) = w \quad .$$

where $w = \sum w_i$ is the total Stiefel-Whitney class of M. Hence the class v_i is some universal polynomial in the w_j. The converse is not true as we soon shall see, as most of the v_i are decomposable. This will be a crucial fact for the main results described in these notes.

Here are some calculations. Let $v = 1 + v_1 + v_2 + \ldots$. We have

$$\begin{aligned}
w_1 &= v_1 \\
Sq^1 v_1 + v_2 &= w_2 \\
Sq^1 v_2 + v_3 &= w_3 \\
Sq^2 v_2 + Sq^1 v_3 + v_4 &= w_4 \\
Sq^2 v_3 + Sq^1 v_4 + v_5 &= w_5 \\
Sq^3 v_3 + Sq^2 v_4 + Sq^1 v_5 + v_6 &= w_6 \\
Sq^3 v_4 + Sq^2 v_5 Sq^1 v_6 + v_7 &= w_7 \\
Sq^4 v_4 + Sq^3 v_5 + Sq^2 v_6 + Sq^1 v_7 + v_8 &= w_8
\end{aligned}$$

Moreover we have the *Wu formula*,

$$Sq^j w_i = \sum_{k=0}^{j} (i-k-1, j-k) w_{i+j-k} w_k$$

For example

$$\begin{aligned}
Sq^1 w_2 &= w_3 + w_2 w_1 \\
Sq^1 w_3 &= w_3 w_1 \\
Sq^2 w_3 &= w_5 + w_4 w_1 + w_3 w_2 \\
Sq^1 w_4 &= w_5 + w_4 w_1
\end{aligned}$$

By solving, we get that v_i is decomposable except if $i = 2^k$, in which case $v_i = w_i$ up to decomposable elements. Here are the low dimensional computations.

$$\begin{aligned}
w_1 &= v_1 \\
w_2 - w_1^2 &= v_2 \\
&= w_2 + \mathrm{dec} \\
v_3 &= w_3 + Sq^1 v_2 \\
&= w_3 + Sq^1 w_2 \\
&= w_3 + w_3 + w_1 w_2 \\
&= w_1 w_2 \\
&= \mathrm{dec} \\
v_4 &= w_4 + w_3 w_1 + w_2^2 + w_1^4 \\
&= w_4 + \mathrm{dec} \\
v_5 &= w_4 w_1 + w_2^2 w_1 + w_3 w_1^2 + w_2 w_1^2 \\
&= \mathrm{dec} \\
v_6 &= \mathrm{dec} \\
v_7 &= \mathrm{dec} \\
v_8 &= w_8 + \mathrm{dec} \\
v_9 &= \mathrm{dec}
\end{aligned}$$

1.3 Computations in a smooth manifold, according to Milnor-Stasheff

Let $M \subset A$ be closed submanifold of codimension k. Let $M \subset M_\epsilon$ be a tubular neighbourhood of this embedding. There is an excision isomorphism

$$H^*(A, A-M) \longrightarrow H^*(N_\epsilon, N_\epsilon - M) \quad .$$

By putting a metric, there is a pair homeomorphism (*the exponential* [BREDON]))

$$(E(\epsilon), E(\epsilon)_0) \longrightarrow (N_\epsilon, N_\epsilon - M)$$

where E is the normal bundle of the embedding, and where for any vector bundle $E \longrightarrow B$, E_0 denotes the total space with the zero section removed. With this metric, we can make sense of D_ϵ the disc bundle of radius ϵ of E, and excision again shows that

$$H^*(A, A-M) \cong H^*(N_\epsilon, N_\epsilon - M) \cong (E(\epsilon), E(\epsilon)_0) \cong H^*(D_\epsilon, (D_\epsilon)_0) \cong H^*(E, E_0).$$

The Thom class $u \in H^k(E, E_0, \mathbf{F}_2)$ yields a class $u' \in H^k(A, A-M, \mathbf{F}_2)$ ($\mathbf{Z}$ coefficients do it in the oriented situation)

Theorem 1.4 *We claim that the map*

$$H^k(A, A-M) \longrightarrow H^k(A) \longrightarrow H^k(M)$$

sends the class u' on the Euler class in the oriented case, and on the top Stiefel-Whitney class in the general case ($\mathbf{F}_2$ coefficients).

Proof. Let $s : M \longrightarrow E$ be the zero section of the normal bundle. Let $\phi : H^k M \longrightarrow H^{2k}(E, E_0)$ be the Thom isomorphism. Then

$$\phi(s^*(u_{|E})) = \pi^*(s^*(u_{|E})) \cup u = u \cup u = Sq^k u$$

and therefore

$$\phi^{-1} Sq^k u = w_k(E) = s^*(u_{|E})$$

The conclusion follows from the commutative diagram:

$$\begin{array}{ccc} H^k(A, A-M) & \longrightarrow & H^k A \\ \downarrow & & \downarrow \\ H^k(N_\epsilon, N_\epsilon - M) & \longrightarrow & H^k(M) \end{array}$$

Definition 1.5 *u' is called the dual class of M in A*

Corollary 1.6 *If M is smoothly embedded as a closed subset of $\mathbf{R}^{n+k}$, then $w_k(\nu^k) = 0$ and $e(\nu^k) = 0$, because, $u' \in H^k(\mathbf{R}^{n+k}) = 0$.*

As $w(\nu^k) = (w(TM))^{-1}$, we get a necessary condition for embeddability.

1.4 The tangent bundle of a manifold

Consider the diagonal embedding $\Delta : M \longrightarrow M \times M$. Then $\nu(\Delta) \cong TM$. The dual class in this case is $u' \in H^n(M \times M, M \times M - \Delta(M))$. We define $j_x : (M, M-x) \longrightarrow (M \times M, M \times M - \Delta(M))$ by $j_x(y) = (x, y)$.

Lemma 1.7 *The class $u' \in H^n(M \times M, M \times M - \Delta(M))$ is uniquely determined by the property that $j_x^*(u')$ is the preferred generator of $H^n M$ for all x.*

There is a restriction homomorphism:

$$H^n(M \times M, M \times M - \Delta(M)) \longrightarrow H^n(M \times M) \quad .$$

1.5 Hirzebruch Signature theorem

Recall that if M^{4k} is a closed manifold, the cup product pairing in the middle dimensional real cohomology evaluated on the orientation class is a non degenerate inner product, and has an associated isomorphism invariant called the signature (difference of positive eingenvalues and negative eigenvalues).

As a matter of fact, the signature is a cobordism invariant, which is easily seen to provide a multiplicative genus, that is, a ring homomorphism

$$\pi_{4*}MSO \longrightarrow \mathbf{Z}$$

By tensoring with $\mathbf{Q}$, Hirzebruch's theory of multiplicative sequences shows that the signature may be recovered in the following way. Consider the power series:

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + (1/3)t - (1/45)t^2 + \ldots + (-1)^{k-1}(2^{2k}B_k/(2k!))t^k + \ldots$$

There is a way to associate to this power series a multiplicative sequence $(L_i)_{i\geq 1}$ called the L-polynomials.

The signature of M^{4k} is the L-polynomial applied to the Pontryagin classes, and evaluated on the fundamental class.

The Pontryagin classes of a real vector bundle are defined using Chern classes. Let $E \longrightarrow B$ be a real vector bundle of dimension n. To this bundle is associated naturally its complexification $E_{\mathbf{C}} \longrightarrow B$, which is a complex vector of dimension $2n$. The chern classes of this new vector bundle are the Pontryagin classes of the former bundle E.

2 Some prerequisites from Stable homotopy

The material of this section is mostly taken from [Browder]

2.1 Stable homotopy of Eilenberg-Mac Lane spaces

K_n denotes an Eilenberg-MacLane space of type $\mathbf{Z}/2\mathbf{Z}$, that is $K_n = K(\mathbf{Z}/2\mathbf{Z}, n)$ is a pointed space of the homotopy type of a CW complex, whose homotopy groups vanish in all degrees except in degree n, and $\pi_n K_n$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$.

Proposition 2.1 $\pi^S_{2n}K_n = \mathbf{Z}/2\mathbf{Z}$

There is a detecting result for the non trivial element $\eta \in \pi_{2n}K_n$.

Proposition 2.2

$$0 \neq \theta \in \pi_{2n}K_n = \eta \Leftrightarrow < Sq^{n+1}, \iota_n, \theta > \neq 0$$

Here, we have $\theta : \Sigma^\infty S^{2n} \longrightarrow \Sigma^\infty K_n$, $\iota_n : \Sigma^\infty K_n \longrightarrow \Sigma^n H\mathbf{Z}/2\mathbf{Z}$ is the adjoint of the canonical homotopy equivalence.

2.2 Steenrod operations and SW duality

Our reference for this part is the unpublished thesis of J. Jones.

Let X be a finite CW-complex. We let X be its Spanier-Whitehead dual (which is actually the function spectrum from X to the sphere spectrum).

There is a map $H_q(X, \mathbf{F}_2) \otimes H^{n-q}(D(X), \mathbf{F}_2) \longrightarrow \mathbf{F}_2$ defined by:

$$f \otimes g \longmapsto \{S^n \wedge X \xrightarrow{f\wedge g} H\mathbf{Z}/2\mathbf{Z} \wedge X \wedge D(X) \longrightarrow \Sigma^n H\mathbf{Z}/2\mathbf{Z}\}$$

This pairing is non degenerate and we get:

Proposition 2.3 *Let X be a compact CW-complex of dimension n. Let $D(X)$ the SW-dual of X. The pairing above is non degenerate induces a isomorphism of graded vector spaces*

$$\Psi : H_*(X, \mathbf{F}_2) \cong H^{*-n}(D(X), \mathbf{F}_2) \quad .$$

Furthermore, the natural right action of the Steenrod algebra $H_*(X, \mathbf{F}_2)$ on translates into the natural left action of the Steenrod algebra on $H^*(D(X), \mathbf{F}_2)$ in the following way. For x in $H_*(X, \mathbf{F}_2)$ and θ in the Steenrod algebra,

$$\Psi(x.\theta) = \Xi(\theta).\Psi(x)$$

where Ξ is the antipode of the Steenrod algebra.

This seems to be well known, but would need a proof in our opinion!

3 Definition and some properties of the Kervaire invariant

All coefficients are $\mathbf{Z}/2\mathbf{Z}$.

3.1 Quadratic forms over $\mathbf{Z}/2\mathbf{Z}$

Let $(-,-)$ be a non degenerate bilinear form over $\mathbf{Z}/2\mathbf{Z}$. A quadratic form relative to $(-,-)$ is a function such that $q(x+y) = q(x) + q(y) + (x, y)$

Theorem 3.1 *Let q be quadratic on the quadratic space E and take any e in E. Then $q_E = q + (e,)$ is again a quadratic form, and all quadratic form are of this sort, in a unique way. In other words, the space of quadratic forms is an affine space directed by E. There are only two isomorphism classes of quadratic forms on a given vector space, classified by the Arf invariant A. There is a formula* $\mathrm{Arf}(q_e) = \mathrm{Arf}(q) + q(e)$.

Remark: Not all non degenerate bilinear form have a quadratic form, for example the dimensions has to be even.

3.2 An interesting quadratic map

This is taken form the book of Browder on surgery on simply connected manifolds.

Assume we have manifolds X and A and a commutative diagram

$$\begin{tikzcd} \nu_X \arrow[r, "f"] \arrow[d] & \eta \arrow[d] \\ X \arrow[r, "f"'] & A \end{tikzcd}$$

We assume that η is trivial. Thom construction induces a map $T(\nu_X) \longrightarrow T(\eta)$ which induces by Atiyah duality a map $g : \Sigma^\infty A_+ \longrightarrow \Sigma^\infty X_+$, hence a map in cohomology $G^* : H^*(X_+) \longrightarrow H^*(A_+)$.

We define a map $\ker g^* \longrightarrow \mathbf{Z}/2\mathbf{Z}$ by the formula

$$x \longmapsto < Sq^{n+1}, x, g > \quad .$$

For A a $2q$-dimensional manifold, and $X = S^{2q}$, and $q = 2k+1$, we get a map $q : H^n X \longrightarrow H^{2n}(A)$. For g degree 1 normal map, we define:

$$\psi(x) = \mathrm{Sq}_g^{q+1}(x)[A] = \mathrm{Sq}_{g\phi_x}^{q+1}(\iota)[A] \quad .$$

Let $x : X \longrightarrow K_q$, $y : X \longrightarrow K_q$ between cohomology classes. We have a stable diagram

$$\begin{tikzcd} A \arrow[r, "g"] & X \arrow[r, "\Delta"] & X \times X \arrow[r, "x \times y"] \arrow[d, "\bar{\Delta}"] & K_q \times K_q \arrow[r, "\mu"] & K_q \arrow[r, "\mathrm{Sq}^{q+1}"] & K_{2q+1} \\ & & \Omega X \vee X \vee (X \wedge X) \arrow[r, "x \vee y \vee x \wedge y"'] & \Omega\, K_q \vee K_q \vee (K_q \wedge K_q) \arrow[u, "\bar{u}"'] & & \end{tikzcd}$$

Hence
$$\psi(x+y) = \psi(x) + \psi(y) + \mathrm{Sq}^{q+1}_{g\bar{\mu}(x\wedge y)\bar{\Delta}}(\iota)$$
but
$$\mathrm{Sq}^{q+1}_{\mu}(\iota) = \Sigma(\iota \wedge \iota)$$
hence
$$<\mathrm{Sq}^{q+1}, \bar{\mu}(x\wedge y)\bar{\Delta}, g> = <\mathrm{Sq}^{q+1}, \iota, \bar{\mu}> (x \wedge y)\bar{\Delta}g = g^*(x \cup y)$$
This proves that ψ is quadratic with respect to the cup product pairing.

3.3 Definition of the Kervaire invariant

Let X be a $2n$-dimensional manifold, whose stable normal bundle is trivial. Then by the above construction, we get a quadratic map whose associated bilinear for is the cup product (evaluated on the fundamental class):
$$\psi_X : H^{2n}X \longrightarrow \mathbf{Z}/2\mathbf{Z}$$
By definition, the Kervaire invariant of the stably framed manifold X is the Arf invariant of ψ_X. We shall see that the Kervaire invariant strongly depends on the framing, as thee are example of different different framing on the same manifold yielding both possible Arf invariants. The Kervaire is a cobordism invariant, hence does not change by framed surgery. We mention, and we sill come back to this later, that the Kervaire invariant is actually a complete obstruction for a framed manifold to be cobordant with a framed homotopy sphere, at least in dimensions $4n+2$ for $n > 0$.

3.4 Cobordism invariance

This is a very sketchy summary of what is well written in Browder's Book!
$$(f, b) : (X, Y) \longrightarrow (A, B)$$

Criterion: $\mathrm{Arf}(\psi) = 0$ if and only if there exists a half dimensional lagrangian $H \subset K^q$ such that $\psi|_H$ is trivial. A pairing is symplectic if it is non singular and there exists a symplectic basis associated to it (that (x_i, y_i) such that $x_i.y_i = 1$, $x_i.x_i = y_i.y_i = 0$). Symplectic forces even dimensional.

There is a diagram
$$\begin{tikzcd}
K^q(X) \arrow[r] \arrow[d] & K^q(Y) \arrow[r] \arrow[d] & K^{q+1}(X,Y) \arrow[d] \\
K_{q+1}(X,Y) \arrow[r] & K_q(Y) \arrow[r] & K_q(X)
\end{tikzcd}$$
vertical maps are Poincaré duality (PD) isomorphisms.

Consider the following situation:$f : V \longrightarrow W$, $f^* : W^* \longrightarrow V^*$ $\ker f = \mathrm{Im}(f^*)$, $V = \ker f \oplus \mathrm{Im} f$ and both have the same rank. Question: $\psi|_H$ trivial ?
$$\begin{tikzcd}
A/B \arrow[r] \arrow[dr] & B \arrow[r] & Y \arrow[r, "\iota"] & X \arrow[r, "x"] & K_q \arrow[r, "\mathrm{Sq}^{q+1}"] & K_{2q+1} \\
& X/Y \arrow[ur] & & & &
\end{tikzcd}$$
Hence $\psi i^* x = 0$

4 Orientations

4.1 Orientations and framings

Let $F \longrightarrow E \longrightarrow B$ be a principal F-fibration with classifying map $B \longrightarrow BF$. Suppose that $f : X \longrightarrow B$ lifts to E. Then the space [liftings] of homotopy classes of liftings is really in bijection with $[X, F]$. There is an action of F on E, and this induces a map

$$[X, F] \times [X, E] \longrightarrow [X, E]$$

That restricts to a map

$$[X, F] \times [\text{liftings}] \longrightarrow [\text{liftings}]$$

that gives [liftings] the structure of an affine space directed by $[X, F]$.

We will essentially consider from now on the case of the fibration $O \longrightarrow EO \longrightarrow BO$, and $K_q \longrightarrow BO < v_q > \longrightarrow BO$ (see the following section).

4.2 Wu orientations and quadratic forms

Consider the universal $(q+1)$-Wu class $v_q : BO \longrightarrow K_{q+1}$. By this we mean that the the Wu classes are universal polynomials in the Stiefel-Whitney classes, and that the universal Stiefel-Whitney classes live in the cohomology of BO, and are therefore classified by maps to Eilenberg-Mac Lane spaces.

Let M be a $2q$ dimensional manifold. The Wu class v_q vanishes by instability. Let $BO < v_q >$ be the homotopy fiber of $V_q : BO \longrightarrow K_{q+1}$, so that we have a principal K_q fibration $K_q \longrightarrow BO < v_q > \longrightarrow BO$ with classifying map $BO \longrightarrow BK_q \simeq K_{q+1}$.

Then the vanishing of the $(q+1)$ Wu class on any $2q$ dimensional manifold M forces the existence of at least a Wu orientation $M \longrightarrow BO < v_q >$.

In fact, it can be shown that there is a quite natural commutative diagram

$$\begin{tikzcd}
O \arrow[r] \arrow[d] & K_q \arrow[d] & \\
EO \arrow[r] \arrow[d] & BO < v_q > \arrow[d] & \\
BO \arrow[r] & BO \arrow[r, "v_q"'] & K_{q+1}
\end{tikzcd}$$

In particular, any framed manifold M has a prefered orientation $M \longrightarrow EO$ hence a canonical Wu orientation $M \longrightarrow EO \longrightarrow BO < v_q >$.

On the other hand, for such manifolds, the Arf invariant was precisely defined in terms of Sq^{q+1} that is represented by multiplication by the $q+1$ Wu class. This link is the reason for the important results that follow.

4.3 Framings and quadratic forms

There is a commutative diagram:

$$\begin{tikzcd}
 & O \arrow[r] \arrow[d] & K_q \arrow[d] & \\
 & EO \arrow[r] \arrow[d] & BO < v_q > \arrow[d] & \\
M \arrow[r] \arrow[ur] & BO \arrow[r] & BO \arrow[r, "v_q"'] & K_{q+1}
\end{tikzcd}$$

It is a calculation that $i : O \longrightarrow K_q$ is trivial if $q \neq 2^i - 1$ and maps to σw_q if $q = 2^i - 1$ and to zero otherwise. This simply comes form the fact that the Wu class is decomposable in the degrees not of the form 2^i, that the map $i : O \longrightarrow K_q$ is the loop map of $v_q : BO \longrightarrow K_{q+1}$, and that looping destroys decomposable elements. Moreover, according to the fact that $v_{2^i} = w_{2^i}$, $i : O \longrightarrow K_q$ represents a non trivil class that is the loop of the Stiefel-Whitney class w_{2^i}.

We already see that something different will happen according to the fact that $2^i = q + 1$ or nor, or in other words, for framed manifolds of dimension $2q = 2^{i+1} - 2$.

Given the data for a change of the framing F on M, that is a map $g : M \longrightarrow O$, we can change the quadratic form ψ_M by the formula

$$\psi'(-) = \psi(-) + < ((gi)^* w_q,) >$$

The question is: how does ψ' relate to the natural quadratic form on the framed manifold $[M, qF]$. This is given by the change of framing formula.

4.4 The Change of framing formula

We now that a framing produces a quadratic form on the middle dimensional cohomology. Given a framing, that is an explicit class $F : M \longrightarrow EO$, we obtain all framings by modifying F along a map $M \longrightarrow O$.

The change of framing formula asserts that:

$$q_{gF}(-) = q_F(-) + < ((gi)^* w_q, -) >$$

In particular, $\mathrm{Arf}(q_{gF}) = \mathrm{Arf}(q_F) + q((gi)^* w_q)$.

In dimensions $n = 2q \neq 2^{i+1} - 2$, we obtain that $\mathrm{Arf}(q_F) = \mathrm{Arf}(q_{gF})$ for all $g : M \longrightarrow O$, that is the quadratic form and its Arf invariant do not depend on the framing.

5 Thom-Pontryagin theory and Kahn-Priddy theorem

5.1 Framed cobordism and stable homotopy

Let X be a topological space and define X^{fr} as the set of map $f : M \longrightarrow X$ where M is a framed manifold, divided by the relation that $f : M \longrightarrow X$ equals $f' : M' \longrightarrow X$ as soon as there is a framed cobordism $B : M \sim M'$ and a map $g : B \longrightarrow X$ restricting to f and f' on the boundary of B.

A parametrized Thom-Pontryagin construction produces a map

$$X^{fr}_n \longrightarrow \pi^S_n X_+$$

In particular $\pi^S_n X$ (without extra base point!) is the subset of X^{fr} where the manifolds are $[M, F]$ are null cobordant.

5.2 Framed and biframed manifolds, and The Kahn-Priddy Theorem

The Kahn-Priddy theorem asserts in its crude form that the transfer $t : \Sigma^\infty \mathbf{R}P^\infty \longrightarrow S^0$ induces an epimorphism of 2 components in positive degrees, that is

$$\text{for } * > 0; \quad {}_{(2)}\pi^S_*(t) : {}_{(2)}\pi^S_* \mathbf{R}P^\infty \longrightarrow {}_{(2)}\pi^S_* S^0$$

is an epimorphism.

Actually, the Segal conjecture computes all maps $\mathbf{R}P^\infty \longrightarrow S^0$ and we have that this group is $\hat{Z}_2$ with unit the transfer. All maps inducing an iso on ${}_{(2)}\pi^S_1$ behave actually like the transfer.

By the Thom-Pontryagin theory, the sequence of maps

$$\mathbf{R}P^\infty \longrightarrow O \longrightarrow S^0$$

really looks like geometrically:

$$[f : M \to \mathbf{R}P^\infty, F : M \to EO] \to [f : M \to O, F : M \to EO] \to [M, F : M \to EO]$$

with (M, F) cobordant to zero.

The surjectivity of ${}_{(2)}\pi_*^S \mathbf{R}P^\infty \longrightarrow {}_{(2)}\pi_*^S S^0$ implies immediately the surjectivity of ${}_{(2)}\pi_*^S O \longrightarrow {}_{(2)}\pi_*^S S^0$, hence *any framed manifold is framed cobordant to a manifold which is itself framed cobordant to zero for some framing.*

In particular, as the kervaire invariant is a framed cobordism invariant, we see that in dimensions $n \neq 2^{i+1} - 2$,

$$\kappa(M, F) = \kappa(M', F)$$

where M' is null cobordant for some framing, but in these dimensions, the Kervaire invariant does precisely not depend on the framing, it must be zero.

6 Adams Spectral sequence interpretetion of the Kervaire invariant one problem

6.1 The Adams tower for S^0

An Adams tower for S^0 is a tower of spectra

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & Y_{n+1} \\
X_{n-1} \arrow[r] \arrow[d] & Y_n \\
X_1 \arrow[r] \arrow[d] & Y_2 \\
S^0 = X_0 \arrow[r] & Y_1
\end{tikzcd}$$

such that

- $X_{n+1} \longrightarrow X_n \longrightarrow Y_{n+1}$ is an exact triangle,
- Y_i is a generalized Eilenberg-Mac Lane spectrum (GEM),
- $H^* f_i$ epimorphism.

All Adams towers are equivalent (in a non canonical way), and any Adams tower immediately provides an Adams spectral sequence, that converge to the p-completed part of the homotopy of S^0.

The existence of such an Adams tower is proved by induction, or one might as well take the canonical resolution associated to the monad of the adjoint pair (forget, $- \wedge H\mathbf{F}_2$), and realize it (totalize).

But what we would like to built is a minimal Adams tower, that is, one for which the E^1-term is actually the E^2-term. Let us see first the existence of such a resolution. Apply H^* any Adams tower. One obtains a sequence of short exact sequences

$$H^{*-1} X_{n+1} \longrightarrow H^* Y_{n+1} \longrightarrow H^* X_n$$

and pasting all these provide a resolution

$$H^* X_0 \longleftarrow H^* Y_1 \longleftarrow H^{*-1} Y_2 \longrightarrow H^{*-2} Y_3 \longleftarrow \dots$$

of $H^* X_0$ by free modules.

Now the spectral sequence is gotten by applying π_* to the Adams tower.

$$\pi_* X_{n+1} \longrightarrow \pi_* X_n \longrightarrow \pi_* Y_n$$

but notice that

$$\pi_* Y_n = \hom^*(H^* Y_n, H^* S^0)$$

and it follows that the E_2-term is precisely $\mathrm{Ext}_{\mathcal{A}_2}(\mathbf{F}_2, \mathbf{F}_2)$. Now, if we take a minimal resolution, that is one such that the number of generators is given by the dimensions given on the E_2-term, then $\hom(d^1, \mathbf{F}_2)$ has to be zero.

Using the information in for example [Kochman], we know how such a minimal resolution has the following Adams tower like:

$$\begin{tikzcd}
X_2 \arrow[r, "f_3"] \arrow[d] & \vee_{i \leq j, i+1 \neq j} \Sigma^{2^i + 2^j - 2} H c_{i,j} \\
X_1 \arrow[r, "f_2"] \arrow[d] & \vee_{i \geq 0} \Sigma^{2^i - 1} H c_i \\
S^0 = X_0 \arrow[r, "f_1"] & H
\end{tikzcd}$$

where H stands for the Eilenberg-Mac Lane spectrum $H\mathbf{Z}/2\mathbf{Z}$.

We have

- $H^* X_1 = \Sigma^{-1} \mathcal{A}_2$
- the map f_2 is the map that sends $c_i \longmapsto \Sigma^{-1} Sq^{2^i}$
- the map f_3 maps $c_{i,i} \longmapsto \sum_{k \leq i} Sq^{2^{i+1} - 2^k} c_k$.

We note that $f_2^* f_3^* c_{i,i} = 0$ because of the Adem relations. The delicate point would be to show that f_3^* is surjective, but we will admit it.

6.2 Explicit computation of differentials

$$\begin{tikzcd}
 & X_n \arrow[r] \arrow[d] & Y_{n+1} \\
 & X_{n-1} \arrow[r] \arrow[d] & Y_n \\
 & X_1 \arrow[r] \arrow[d] & Y_2 \\
S^t \arrow[r, "\theta"'] & S^0 = X_0 \arrow[r] & Y_1
\end{tikzcd}$$

Assume $f_1 \theta \neq 0$. This forces $t = 0$ and the degree of f has to be odd.

Now if $f_1\theta = 0$, this implies that f lifts to X_1 as a map θ_1.

$$\begin{tikzcd}
 & X_n \arrow[r, "f_{n+1}"] \arrow[d] & Y_{n+1} \\
 & X_{n-1} \arrow[r, "f_n"] \arrow[d, dotted, no head] & Y_n \\
 & X_1 \arrow[r, "f_2"] \arrow[d] & Y_2 \\
S^t \arrow[r, "\theta"'] \arrow[ur, "\theta_1"] & S^0 = X_0 \arrow[r, "f_1"] & Y_1
\end{tikzcd}$$

Now $f_i\theta_1 \neq 0$ forces $t = 2^i - 1$. Moreover, if we let $p_i : \vee_{i\geq 0}\Sigma^{2^i-1}Hc_i \longrightarrow \Sigma^{2^i-1}Hc_i$ be the obvious projection, and g be the composition $Y_1 \longrightarrow \Sigma X_1 \longrightarrow \Sigma Y_2 \longrightarrow \Sigma^{2^i}Hc_i$ where the last map is Σp_i, then the composition $p_i f_2\theta_1$ belongs to $< g, f_1, \theta >$ with zero indeterminacy and is non zero.

More, in this case, θ is detected by h_i, and the cone of Θ supports a primary operation. That is the map θ has Hopf invariant one.

If we begin with θ_1, we see by a similar reasoning that θ is detected by h_i^2 if and only the Toda bracket $< g, f_2, \theta_1 >$ is non zero (modulo zero), and that this condition corresponds to the cone of θ supporting the secondary operation corresponding to the Adem relation defining h_i^2. Here the map g is the composition:

$$Y_2 \longrightarrow \Sigma X_2 \longrightarrow \vee_{i\leq j, i+1\neq j}\Sigma^{2^i+2^j-1}Hc_{i,j} \longrightarrow \Sigma^{2^i+2^i-1}Hc_{i,i} \ .$$

We use implicitly that in these degrees, a permanent cycle can not be a boundary.

6.3 Refinement of the detection result

By the Kahn Priddy theorem, the map θ_1 factorizes through $\mathbf{R}P^\infty$

$$\begin{tikzcd}
S^{2^{i+1}-2} \arrow[rr, "\theta_1"] \arrow[dr, "\lambda"'] & & X_1 \\
 & \mathbf{R}P^\infty \arrow[ur, "f"'] &
\end{tikzcd}$$

hence:

$$\begin{aligned}
\theta_1 \text{ detected by } h_i^2 \quad &\Leftrightarrow \quad < \phi_i, c, \theta_1 > \neq 0 \\
&\Leftrightarrow \quad < \phi_i, c, f\lambda > \neq 0 \\
&\Leftrightarrow \quad < \phi_i, cf, \lambda > \neq 0
\end{aligned}$$

Lemma 6.1

$$f_*c_t = u^{2^t-1}$$

Hence the situation is:

$$S^{2^i-2} \xrightarrow{\lambda} \mathbf{R}P^\infty \xrightarrow{u^{(2^t-1)_{t\leq i}}} \vee_{t\leq i}\Sigma^{2^t-1}H \xrightarrow{\phi_i} \Sigma^{2^{i+1}-1}H$$

By instability, $Sq^{2^{i+1}-2^t}u^{2^t-1} = 0$ for $t \leq i$, hence the Toda brackets $< Sq^{2^{i+1}-2^t}, u^{2^t-1}, \lambda >$ are defined separately. An elementary exercise shows that:

$$< \phi_i, (u^{2^t-1})_{t\leq i}, \lambda > = \sum_{0\leq t\leq i} < Sq^{2^{i+1}-2^t}, u^{2^t-1}, \lambda >$$

Now we have:

Proposition 6.2 *For $0 \leq t \leq i$, $< Sq^{2^{i+1}-2^t}, u^{2^t-1}, \lambda > = 0$*

This implies immediately that:

Proposition 6.3 *θ_1 detected by h_i^2 if and only if $< Sq^{n+1}, u^n, \lambda >$ is non zero.*

6.4 Detecting criterion for manifolds of Kervaire invariant 1

Assume there is a framed manifold $[M, F]$ of Kervaire invariant 1, then there is a framed cobordant manifold $[N, F']$ of Kervaire invariant one who has Kervaire zero for some framing G.

Hence the difference of framings $g = G/F : M \longrightarrow O$ is a map such that $q_F(g^*\Omega v_{n+1}) = 1$ Now we recall how one computes q_F.

Given a cohomology class $u : M \longrightarrow K_n$, $q_F(u)$ is the element of $\pi_{2n}K_n$

$$S^{2n+k} \longrightarrow T\nu \longrightarrow T\nu \wedge M_+ \xrightarrow[f\wedge a]{} S^k \wedge K_n$$

where ν is the stable normal bundle of M. By the Kahn-Priddy theorem, there is a commutative diagram

$$\begin{tikzcd}
S^{2n+k} \arrow[r] \arrow[ddrr] & T\nu \wedge M_+ \arrow[r] & S^k M \arrow[rr] \arrow[dd] \arrow[dr] & & S^k K_n \arrow[r] & \Sigma^k H_n \arrow[r] & H_{2n+1} \\
& & & S^k \wedge O \arrow[ur] & & & \\
& & S^k\mathbf{R}P^\infty \arrow[ur] \arrow[uurr, bend right] & & & &
\end{tikzcd}$$

and we check easily that

$$\begin{aligned}
[M, gF] \text{ has Kervaire invariant } 1 \quad &\Leftrightarrow \quad < Sq^{n+1}, \iota_n, q_F g^*(\Omega v_{n+1}) >= 1 \\
&\Leftrightarrow \quad < Sq^{n+1, \iota_n, u^n\lambda} >= 1 \\
&\Leftrightarrow \quad < Sq^{n+1}, u^n, \lambda >= 1 \\
&\Leftrightarrow \quad [M, gF] \text{ is detected by } h_i^2
\end{aligned}$$

7 The Kervaire invariant problem: approaches and results

Important Remark: P. Akhmetiev has recently posted a preprint where he seems to prove that there are no Kervaire invariant one elements in high degrees.

7.1 The Kervaire invariant problem

A homotopy sphere (*h-sphere*) is a smooth manifold homotopy equivalent to the ordinary sphere. The set Θ^n of h-cobordism classes of oriented h-spheres of dimension n is an Abelian group under connected sum. The h-cobordism theorem allows us to equate Θ^n with *the set of smooth structures on* S^n provided $n \geq 5$. Using the Thom-Pontryagin construction, one establishes an exact sequence

$$0 \longrightarrow bP^{n+1} \longrightarrow \Theta^n \xrightarrow{K} \pi_n^S/\mathrm{Im}J_n = \mathrm{coker}J_n$$

where bP^{n+1} is the group of oriented h-spheres of dimension n bounding a parallelizable manifold, π_n^S is the n^{th} stable homotopy group of spheres and $J_n : \pi_n O \longrightarrow \pi_n^S$ is the J-homomorphism from the homotopy groups of the infinite orthogonal group to π_n^S. The map K is defined because any homotopy sphere is stably parallelizable.

This sequence is analyzed with the help of surgery theory and one shows that the map $K : \Theta^n \longrightarrow \mathrm{coker}J_n$ is at most $\mathbf{Z}/2$ in dimensions $n = 4k + 2$. This $\mathbf{Z}/2$ indeterminacy corresponds to a surgery obstruction -*the Kervaire invariant*- and the realizability of this obstruction by a manifold is called the *Kervaire invariant* 1 *problem.*

We proved, following Jones' method, the fundamental result due to Browder:

1. the Kervaire obstruction vanishes in dimensions $n = 4k + 2 \neq 2^{i+1} - 2$,

2. there exists a manifold of Kervaire invariant 1 in dimension $2^{i+1} - 2$ if and only if there exists a stable map θ_i detected by the class h_i^2 in the classical mod 2 Adams spectral sequence (in the Adams spectral sequence language : there are maps of Adams filtration 2 detected by h_i^2).

That is, a framed manifold $[M, F]$ defines a non trivial element in coker$T$ if and only if $\kappa([M, F]) = 1$, and this happens if and only if there is not framed cobordant to any homotopy sphere. Has to manifolds with the same Kervaire invariant are immediately framed cobordant [reference], we obtain that coker$T$ is at most $\mathbf{Z}/2\mathbf{Z}$, and that this happens only in the case where $n =^{i+1} -2$ and h_i^2 survives in the ASS. We notice that the relation is established in such a way that a framed manifold detected by h_i^2 would have precisely Kervaire invariant 1.

Now, we noticed that the only (except for finitely many cases) possible permanent cycles on the second line of the ASS are h_i^2 and $h_1 h_i$. The family $h_1 h_i$ was shown by Mahowald to be a permanent family. The question for the h_i^2 family remains open at the time of this writing, and we would like to discuss a little the way things are going to work.

7.2 Transfer complexity

We heavily used the transfer map $t : \mathbf{R}P^\infty \longrightarrow S^0$ in conjunction with the Kahn-Priddy theorem. One can iterate the construction in the following manner.

$$\mathbf{R}P^\infty \wedge \mathbf{R}P^\infty \xrightarrow{t \wedge t} S^0 \wedge S^0 \longrightarrow S^0$$

Working this out a little bit, one gets a transfer tower and a map from this tower to the Adams Tower.

The tranfer is a map $t : \mathbf{R}P^\infty \longrightarrow S^0$. We consider the fiber sequence

$$X_1 \longrightarrow S^0 \longrightarrow H\mathbf{Z}/2\mathbf{Z}$$

It is well known that t is trivial on the zeroth homology group, hence the composition

$$\mathbf{R}P^\infty \longrightarrow S^0 \longrightarrow H\mathbf{Z}/2\mathbf{Z}$$

is trivial, and this in turn implies that t has a lifting $t' : \mathbf{R}P^\infty \longrightarrow X_1$.

There is now a fiber sequence:

$$X_2 \longrightarrow X_1 \longrightarrow H\mathbf{Z}/2\mathbf{Z} \wedge X_1$$

The map $t \wedge t : \mathbf{R}P^\infty \wedge \mathbf{R}P^\infty \longrightarrow S^0$ factorizes as

$$\mathbf{R}P^\infty \wedge \mathbf{R}P^\infty \xrightarrow{t \wedge 1} \mathbf{R}P^\infty \xrightarrow{t} S^0$$

and $t \wedge t$ is the composition:

$$\mathbf{R}P^\infty \wedge \mathbf{R}P^\infty \xrightarrow{t \wedge 1} \mathbf{R}P^\infty \xrightarrow{t'} X_1 \longrightarrow S^0$$

The composition

$$\mathbf{R}P^\infty \wedge \mathbf{R}P^\infty \xrightarrow{t \wedge 1} \mathbf{R}P^\infty \xrightarrow{t'} X_1$$

is trivial in homology because t is, and we get a lifting $(t \wedge t)' : \mathbf{R}P^\infty \wedge \mathbf{R}P^\infty \longrightarrow X_2$.

An induction allow one to construct a map of towers:

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] \\
\mathbf{R}P^{\infty\wedge 3} \arrow[r, "\varphi_3"] \arrow[d] & X_3 \arrow[d] \\
\mathbf{R}P^{\infty\wedge 2} \arrow[r, "\varphi_2"] \arrow[d] & X_2 \arrow[d] \\
\mathbf{R}P^{\infty} \arrow[r, "\varphi_2"] \arrow[d] & X_1 \arrow[d] \\
S^0 \arrow[r, "="] & S^0
\end{tikzcd}$$

The transfer is a map $t : \mathbf{R}P^\infty \longrightarrow S^0$. We consider the fiber sequence

$$X_1 \longrightarrow S^0 \longrightarrow H\mathbf{Z}/2\mathbf{Z}$$

It is well known that t is trivial on the zeroth homology group, hence the composition

$$\mathbf{R}P^\infty \longrightarrow S^0 \longrightarrow H\mathbf{Z}/2\mathbf{Z}$$

is trivial, and this in turn implies that t has a lifting $t' : \mathbf{R}P^\infty \longrightarrow X_1$. There is now a fiber sequence:

$$X_2 \longrightarrow X_1 \longrightarrow H\mathbf{Z}/2\mathbf{Z} \wedge X_1$$

The map $t \wedge t : \mathbf{R}P^\infty \wedge \mathbf{R}P^\infty \longrightarrow S^0$ factorizes as

$$\mathbf{R}P^\infty \wedge \mathbf{R}P^\infty \xrightarrow{t\wedge 1} \mathbf{R}P^\infty \xrightarrow{t} S^0$$

and $t \wedge t$ is the composition:

$$\mathbf{R}P^\infty \wedge \mathbf{R}P^\infty \xrightarrow{t\wedge 1} \mathbf{R}P^\infty \xrightarrow{t'} X_1 \longrightarrow S^0$$

The composition

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An induction allow one to construct a map of towers:

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\mathbf{R}P^{\infty\wedge 2} \arrow[r, "\varphi_2"] \arrow[d] & X_2 \arrow[d] \\
\mathbf{R}P^{\infty} \arrow[r, "\varphi_2"] \arrow[d] & X_1 \arrow[d] \\
S^0 \arrow[r, "="] & S^0
\end{tikzcd}$$

One gets the transfer conjecture: any map $\mathbf{R}P^\infty \longrightarrow S^0$ lifts to a certain transfer filtration, and once this is done, the map is detected by a spherical element.

Recall that the Whitney sum of bundles induces a map $BO \times BO \longrightarrow BO$. The effect of looping this map is a map $w : O \times O \longrightarrow O$, which we call the Whitney map. Recall also the fundamental diagram:

$$\begin{tikzcd}
 & O \arrow[r] \arrow[d] & K_n \arrow[d] & \\
 & EO \arrow[r] \arrow[d] & BO < v_{n+1} > \arrow[d] & \\
M \arrow[r] \arrow[ur] \arrow[uur] & BO \arrow[r] & BO \arrow[r] & K_{n+1}
\end{tikzcd}$$

If $g : M \longrightarrow O$ is a map, we are interested in $g^* y_n$, where y_n is the image of the Stiefel-Whitney class w_{n+1} through the cohomology suspension. Recall also that if $s : \mathbf{R}P^\infty \longrightarrow O$ is the real reflection map, $s^* y_n$ is u^n.

Theorem 7.1 $[M^{2n}, F]$ *a framed manifold and* $g_1, g_2 \longrightarrow O$ *two maps. Then* $w(g_1, g_2)^* y_n = g_1^* y_n + g_2^* y_n$.

Indeed, being in the image of the cohomology suspension, y_n is primitive (for w^*), and we get

$$w(g_1, g_2)^* y_n = \Delta^*(g_1^*, g_2^*) w^*(y_n) = \Delta^*(g_1^*, g_2^*)(1 \otimes y_n + y_n \otimes 1) = g_1^* y_n + g_2^* y_n$$

Now assume that $[M^{2n}, F]$ a framed manifold with $g_1, g_2 : M \longrightarrow O$, $g_1^* y_n \cup g_2^* y_n = 1$. Then it follows that ...

Assume that F is a framing of Kervaire invariant zero. We now that for a general $g : M \longrightarrow O$,

$$\mathrm{Arf}(q_{gF}) = q_F(g^* y_n)$$

hence

$$\begin{aligned} q_F(-) &= q_{(w(g_1,g_2))F} + < (w(g_1, g_2))F^* y_n, - > \\ &= q_{(w(g_1,g_2))F} + < (g_1^* y_n + g_2^* y_n, - > \\ &= q_{(w(g_1,g_2))F} + < g_1^* y_n, - > + < g_2^* y_n, - > \end{aligned}$$

and

$$\begin{aligned} \mathrm{Arf}(q_{(w(g_1,g_2))F}) &= q_F((w(g_1, g_2))F^* y_n) \\ &= q_F(g_1^* y_n + g_2^* y_n) \\ &= q_F(g_1^* y_n) + q_F(g_2^* y_n) + g_1^* y_n \cup g_2^* y_n \\ &= q_F(g_1^* y_n) + q_F(g_2^* y_n) + g_1^* y_n + 1 \end{aligned}$$

Hence one among $q_F(g_1^* y_n)$, $q_F((w(g_1, g_2))F^* y_n)$, $q_F(g_2^* y_n)$ is one, hence M has a framing of Kervaire invariant one. It is not difficult to see that elements of Kervaire invariant one are detected in this fashion if and only if they factor through the double transfer.

The same argument shows that η^2, ν^2, σ^2, the squares of Hopf maps do factor through the double transfer. Minami proved:

Theorem 7.2 *θ_i if it exists, does not factor through the double transfer for $i \geq 5$*

Lin and Mahowald completed the answer and proved

Theorem 7.3 *θ_i does factor through the double transfer for $i \leq 4$.*

Hence the transfer complexity of the Kervaire family, if it exists is maximal: The Kervaire elements won't be decomposable in the sense of the transfer.

7.3 Structure of the family

Now, should the family θ_i exist?

There are inductive approaches to the construction of the θ_i-family. One of them relies on:

Theorem 7.4 *If θ_i exists such that $2\theta_i = 0$ and $\theta_i^2 = 0$, then θ_{i+1} exists.*

This is of course not really an induction, because by constructing θ_{i+1} in this fashion, it is less than obvious that θ_{i+1} will satisfy the induction property. The elements θ_4 and θ_5 are constructed by this inductive approach. This makes one think that if θ_i fails to exist for some i, then θ_k will fail to exist for all $k > i$.

We notice that $Sq^0 h_i^2 = h_{i+1}^2$, and the Sq^0 on the E^2-term converges to the Root invariant. It follows that if θ_i and θ_{i+1} exist, then θ_{i+1} is in the root invariant of θ_i.

7.4 The situation

$$\begin{array}{c}\text{Have a homotopy theoretic}\\ \text{construction}\end{array} \left\{ \begin{array}{c} \text{have a geometric construction} \\ \text{and are detected} \\ \text{by the double tranfer} \\ \\ \text{are not detected} \\ \text{by the double transfer} \end{array} \right. \begin{array}{l} \left\{ \begin{array}{l} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{array} \right. \\ \left\{ \begin{array}{l} \theta_5 \\ \vdots \end{array} \right. \end{array}$$

7.5 Argments for the existence

Assuming that conjectural picture, there should exist an index $k \geq 5$ such that θ_i exists for for all $i \leq k$ and fails to for $i > k$. The question is of course if k is infinite.

A well known trick shows that h_i^2 survives and detects a map θ_i *of order* 2 yields the existence of η_{j+1}. All the known maps of Kervaire invariant 1 $\theta_i, i \leq 5$ can be chosen of order 2.

Thinking of the existence of η_j as a manifestation of the existence of θ_i, this would suggest that k is infinite. This is so to say the optimistic hypothesis.

7.6 Arguments against the existence

We might also draw a parallel between the Hopf family on the 1-line and the Kervaire family on the 2-line of the ASS.

First, the maps of hops invariant one η, ν, σ are detected by h_1, h_2 and h_3, which are the only permanent cycles on the first line of the ASS. They are all in the image of the J-homomorphism, and detect the existence of division algebra structures on euclidian spaces. On the E^2 term of the ASS, these maps are related by Sq^0 operation, hence the root invariant of η contains ν and the root invariant of ν contains σ. The complexity of the family increases in the sense that η reflects the existence of a commutative associative division algebra structure, while ν reflects the existence of non commutative associative division algebra structure, and σ reflects the existence of a neither associative nor commutative division algebra structure.

Finally, the Adams differential

$$d_2 h_i = h_0 h_{i-1}^2$$

holds for $i \geq 2$ but is zero for $i = 2, 3$, and this explains why the h_2 and h_3 are infinite cycles.

$$[h_i h_{i+1} = 0, h_i h_{i+1}^2 = 0 h_i^2 h_{i+2} = h_{i+3}]$$

Now, a well known calculation shows that the root invariant of σ is η^2 (reference?).

This leads one to suspect that, reflecting the increasing complexity of the already constructed θ_i, there could be some differential pattern that are non trivial only in dimensions $i >> 0$, that would kill the Kervaire elements. Such differential pattern should fit in the scheme given by the *new doomsday conjecture*, that says that the Root invariant has to eventually increase the Adams filtration upon iteration.

Another reason to suspect that k is finite is that θ_4 was shown by Milgram to be detected in the cohomology of some sporadic group. Hence the sparsity of the Kervaire family might be related to the finiteness of the family of Sporadic simple groups.