Decimation-in-frequency Fast Fourier Transforms for the Symmetric Group

Eric Malm

Presentation Days '05
Thesis Advisor: Michael Orrison
Harvey Mudd College

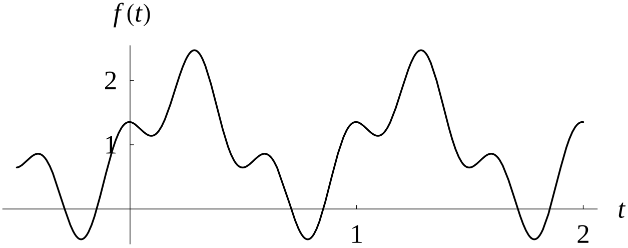
02 May 2005

Overview

Signal Analysis

The Setup

Suppose we want to analyze some periodic signal f

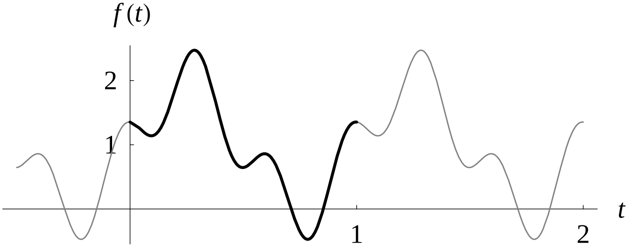


Signal Analysis

The Setup

Suppose we want to analyze some periodic signal f

- Pick some full time period of f

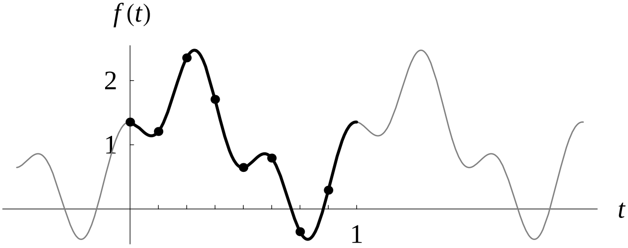


Signal Analysis

The Setup

Suppose we want to analyze some periodic signal f

- Pick some full time period of f
- Take N samples $f_0, f_1, \dots, f_{N-1}$ of f in this time period



Discrete Fourier Transforms

$$\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix}$$

- Process $f_0, \ldots, f_{N-1}$ with the Discrete Fourier Transform

Discrete Fourier Transforms

$$\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} \xrightarrow{\text{input}} \boxed{\hat{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-2\pi ijk/N}}$$

- Process $f_0, \ldots, f_{N-1}$ with the Discrete Fourier Transform

Discrete Fourier Transforms

$$\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} \xrightarrow{\text{input}} \boxed{\hat{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-2\pi ijk/N}} \xrightarrow{\text{output}} \begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{N-1} \end{pmatrix}$$

- Process $f_0, \ldots, f_{N-1}$ with the Discrete Fourier Transform
- Get N complex numbers $\hat{f}_0, \ldots \hat{f}_{N-1}$ such that

$$f(t) \approx \sum_{k=0}^{N-1} \quad \hat{f}_k \quad \left(\cos\frac{2\pi k}{N}t + i\sin\frac{2\pi k}{N}t\right)$$

Discrete Fourier Transforms

$$\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} \xrightarrow{\text{input}} \boxed{\hat{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-2\pi ijk/N}} \xrightarrow{\text{output}} \begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{N-1} \end{pmatrix}$$

- Process $f_0, \ldots, f_{N-1}$ with the Discrete Fourier Transform
- Get N complex numbers $\hat{f}_0, \ldots \hat{f}_{N-1}$ such that

$$f(t) \approx \sum_{k=0}^{N-1} \hat{f}_k \left(\cos \frac{2\pi k}{N} t + i \sin \frac{2\pi k}{N} t\right)$$

"pure" frequency

Discrete Fourier Transforms

$$\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{pmatrix} \xrightarrow{\text{input}} \boxed{\hat{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-2\pi i jk/N}} \xrightarrow{\text{output}} \begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{N-1} \end{pmatrix}$$

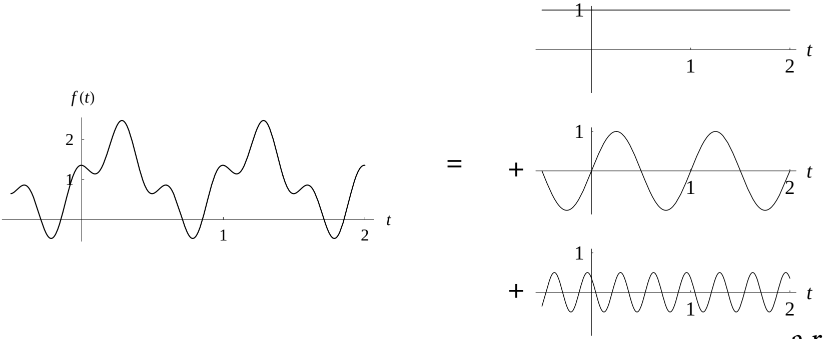
- Process $f_0, \ldots, f_{N-1}$ with the Discrete Fourier Transform
- Get N complex numbers $\hat{f}_0, \ldots \hat{f}_{N-1}$ such that

$$f(t) \approx \sum_{k=0}^{N-1} \underbrace{\hat{f}_k}_{\text{amplitude}} \left(\cos \frac{2\pi k}{N} t + i \sin \frac{2\pi k}{N} t\right)$$

Example

Example

Our original signal is secretly the sum of three "pure" frequencies:



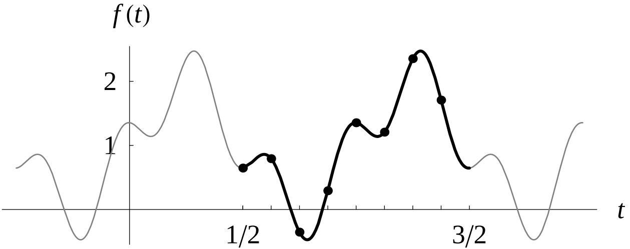
Significance of the DFT

Time-Shift Invariance

Suppose we sampled our signal f over a *different* time period

- The samples $f_0, \ldots, f_{N-1}$ could be much different
- But the Fourier coefficients $\hat{f}_0, \ldots, \hat{f}_{N-1}$ will not be

The DFT is therefore invariant under translational symmetry

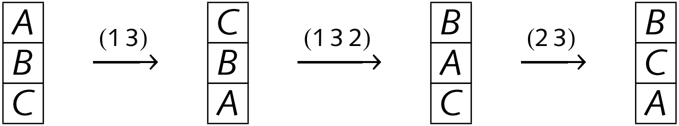


Symmetries and Groups

Symmetries and Groups

- Different spaces have different symmetries

Space	Symmetry
time domain	time translations
sphere	rotations
lists	permutations

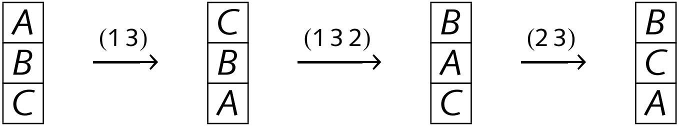


Symmetries and Groups

Symmetries and Groups

- Different spaces have different symmetries
- Write symmetries abstractly as groups

Space	Symmetry	Group
time domain	time translations	$\mathbb{Z}/N\mathbb{Z}$
sphere	rotations	$SO(3)$
lists	permutations	S_n

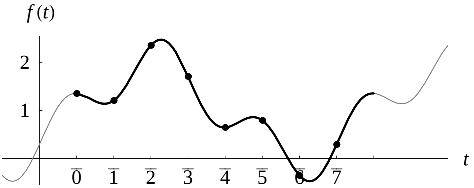


Group Algebras

Reformulation as Group Algebra

- Treat functions on spaces as functions on groups
- Rewrite functions on group as group algebra elements:

$$f : X \to \mathbb{C} \longrightarrow f : G \to \mathbb{C} \longrightarrow \sum_{g \in G} f(g)\, g$$



Wedderburn's Theorem

Theorem (Wedderburn)

The group algebra $\mathbb{C}G$ of a finite group G is isomorphic to an algebra of block diagonal matrices:

$$\mathbb{C}G \cong \bigoplus_{j=1}^{h} \mathbb{C}^{d_j \times d_j}$$

Example ($\mathbb{C}S_3$)

$$\mathbb{C}S_3 \cong \mathbb{C}^{1\times 1} \oplus \mathbb{C}^{2\times 2} \oplus \mathbb{C}^{1\times 1} = \begin{pmatrix} \bullet & & & \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ & & & \bullet \end{pmatrix}$$

Generalized Discrete Fourier Transforms (DFTs)

Definition (Generalized DFT)

Any such isomorphism D on $\mathbb{C}G$ is a generalized DFT for G

- Coefficients in matrix $D(f)$: generalized Fourier coefficients
- Blocks along diagonal: smallest $\mathbb{C}G$-invariant spaces in $\mathbb{C}G$

Change of Basis

DFT a change of basis into a symmetry-invariant basis

- Picking standard bases on $\mathbb{C}G$, matrix algebra gives DFT matrix
- Naïve bound of $O(|G|^2)$ on complexity of DFT evaluation

Decimation-In-Frequency Fast Fourier Transforms (FFTs)

Decimation-In-Frequency FFT

- Fix chain of subgroups of G:

$$1 = G_0 < G_1 < \cdots < G_{n-1} < G_n = G.$$

- Project into successively smaller subspaces in stages corresponding to subgroups
- Goal: Obtain sparse factorization of change-of-basis matrix D

S_n an Ideal Proof-of-Concept Group

Nonabelian, representation theory well understood, natural chain of subgroups

$$1 < S_2 < S_3 < \cdots < S_n$$

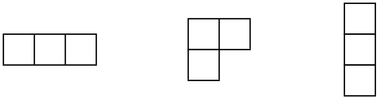
Representation Theory of S_n

Each block in matrix algebra for $\mathbb{C}S_n$ corresponds to a different non-increasing (proper) partition of n

Example ($\mathbb{C}S_3$)

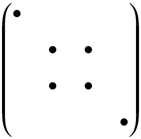
$$\mathbb{C}S_3 \cong \begin{pmatrix} \bullet \end{pmatrix} \oplus \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix} \oplus \begin{pmatrix} \bullet \end{pmatrix}$$

$3 \mapsto$

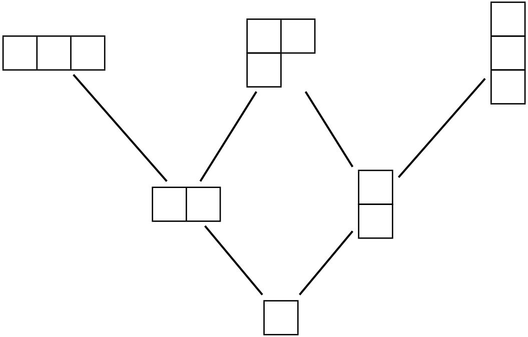


Character Graph for $1 < S_2 < S_3$

Graded diagram of proper partitions for $1 < S_2 < \cdots < S_n$

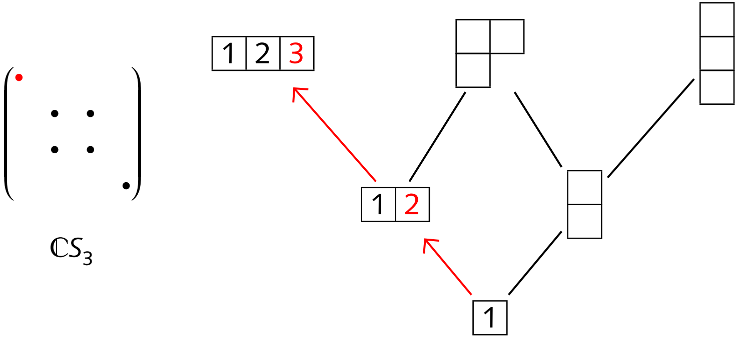


$\mathbb{C}S_3$



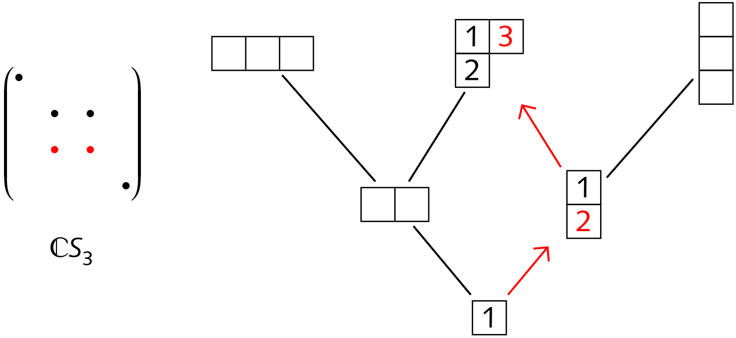
Character Graph for $1 < S_2 < S_3$

Each pathway through the diagram corresponds to a filled-in partition and a row/column in matrix



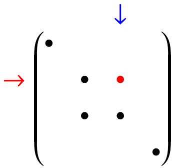
Character Graph for $1 < S_2 < S_3$

Each pathway through the diagram corresponds to a filled-in partition and a row/column in matrix

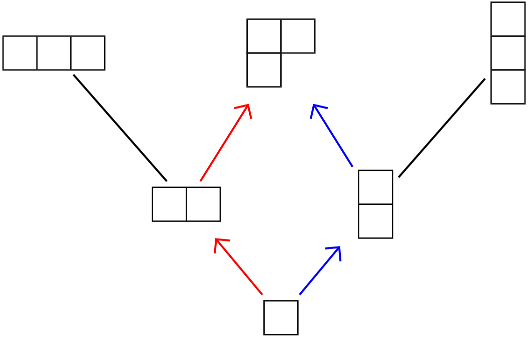


Character Graph for $1 < S_2 < S_3$

A pair of paths ending at same diagram specifies a 1-D Fourier space



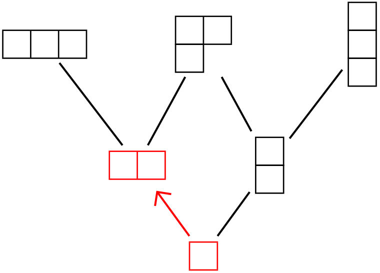
$\mathbb{C}S_3$



Construction of Factorization

Stages of Subspace Projections

- Partial paths give $\mathbb{C}S_n$ subspaces
- At stage for S_k, project onto these subspaces
- Build sparse factor from projections
- Full paths by stage for S_n

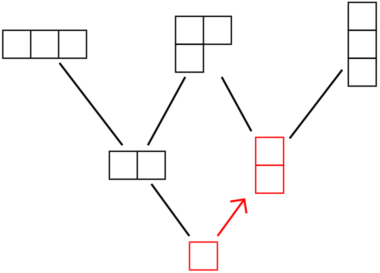


$$\mathbb{C}S_3 \cong \left(\cdot\right) \oplus \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \oplus \left(\cdot\right)$$

Construction of Factorization

Stages of Subspace Projections

- Partial paths give $\mathbb{C}S_n$ subspaces
- At stage for S_k, project onto these subspaces
- Build sparse factor from projections
- Full paths by stage for S_n



$$\mathbb{C}S_3 \cong \left(\bullet\right) \oplus \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix} \oplus \left(\bullet\right)$$

Factorization of $\mathbb{C}S_3$ DFT Matrix

Full DFT matrix for $\mathbb{C}S_3$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{3}{2} & \frac{3}{2} & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

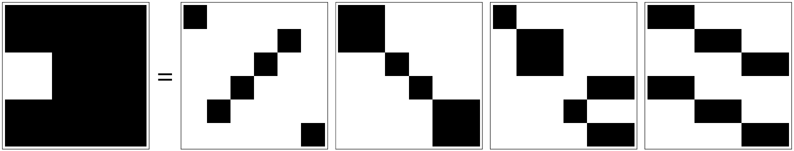
Factorization of $\mathbb{C}S_3$ DFT Matrix

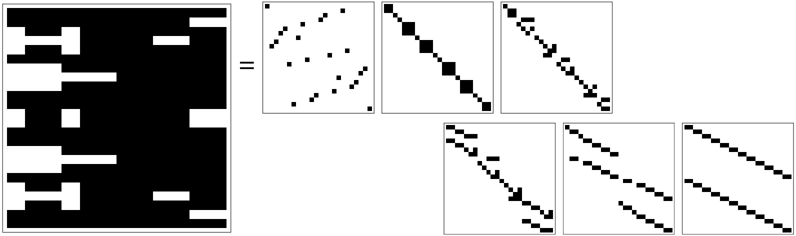
Three factors ($\mathbb{C}S_2$ on rows, $\mathbb{C}S_2$ on columns, $\mathbb{C}S_3$ on rows):

$$\begin{pmatrix} 1 & 1 & & & & \\ 1 & -\frac{1}{2} & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \frac{1}{2} \\ & & & & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ & 1 & 1 & & & \\ & 1 & -1 & & & \\ & & & & 1 & -1 \\ & & & 1 & & \\ & & & & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & & & 1 & 1 \\ 1 & -1 & & & & \\ & & 1 & -1 & & \\ & & & & 1 & -1 \end{pmatrix}$$

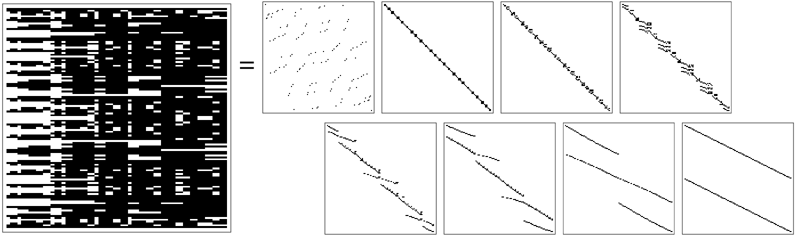
plus a permutation matrix and a row-scaling diagonal matrix

Factorization of $\mathbb{C}S_n$ DFT Matrices





Factorization of $\mathbb{C}S_n$ DFT Matrices



Results from Prototype Implementation

Prototype *MATHEMATICA®* Implementation

Can compute exact FFT up to $n = 6$

Operation Counts For Evaluation

n	t_n^{full}	t_n^{DIF}	t_n^{Maslen} [1]	$\frac{1}{2}n(n-1)$
3	4.7	2.7	2.7	3
4	18.8	5.3	5.4	6
5	87.9	8.8	9.1	10
6	486.4	13.8	13.6	15

Future Directions

Theory

- Prove $O(n^2|S_n|)$ bounds on operation counts
- Deduce better bases for blocks in factors
- Relate FFT on S_n to FFTs on S_{n-1}

Implementation

- Improve efficiency of *MATHEMATICA*® implementation
- Port to MATLAB or GAP
- Parallelize decimation-in-frequency FFT algorithm

References

[1] David K. Maslen.
The efficient computation of fourier transforms on the symmetric group.
Mathematics of Computation, 67(223):1121–1147, July 1998.

[2] David K. Maslen and Daniel N. Rockmore.
Generalized FFTs - a survey of some recent results.
DIMACS Series in Discrete Mathematics and Computer Science, 28:183–237, 1997.

[3] Bruce E. Sagan.
The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions.
Wadsworth & Brooks/Cole, Pacific Grove, CA, 1991.

On The Web

Senior thesis website: ⟨http://www.math.hmc.edu/~emalm/thesis/⟩