STRING TOPOLOGY AND THE BASED LOOP SPACE

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Abstract

This thesis focuses primarily on understanding some of the structures of the string topology of a manifold through homotopy-theoretic constructions on the based loop space of the manifold. In their seminal work on string topology, Chas and Sullivan showed that, for M a closed, oriented manifold, the homology of its free loop space, LM, forms a Batalin-Vilkovisky (BV) algebra under the loop product and the loop-rotation operator Δ. We relate this structure to the homological algebra of the singular chains $C_*\Omega M$ of the based loop space of M, showing that its Hochschild cohomology $HH^*(C_*\Omega M)$ carries a BV algebra structure isomorphic to that of string topology. Furthermore, this structure is compatible with the usual cup product and Lie bracket on Hochschild cohomology. This isomorphism arises from a derived form of Poincaré duality using $C_*\Omega M$-modules as local coefficient systems. This derived Poincaré duality also comes from a form of fibrewise Atiyah duality on the level of fibrewise spectra, and we use this perspective to connect the algebraic constructions to the Chas-Sullivan loop product.

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Contents

Chapter 1

Introduction

String topology, as initiated by Chas and Sullivan in their 1999 paper [4], is the study of algebraic operations on $H_*(LM)$, where M is a closed, smooth, oriented d-manifold and $LM = \mathrm{Map}(S^1, M)$ is its space of free loops. They show that, because LM fibers over M with fiber the based loop space ΩM of M, $H_*(LM)$ admits a graded-commutative *loop product*

$$\circ : H_p(LM) \otimes H_q(LM) \to H_{p+q-d}(LM)$$

of degree $-d$. Geometrically, this loop product arises from combining the intersection product on $H_*(M)$ and the Pontryagin or concatenation product on $H_*(\Omega M)$. Writing $\mathbb{H}_*(LM) = H_{*+d}(LM)$ to regrade $H_*(LM)$, the loop product makes $\mathbb{H}_*(LM)$ a graded-commutative algebra. Chas and Sullivan describe this loop product on chains in LM, but because of transversality issues they are not able to construct the loop product on all of C_*LM this way. Cohen and Jones instead give a homotopy-theoretic description of the loop product in terms of a ring spectrum structure on a generalized Thom spectrum LM^{-TM}.

$H_*(LM)$ also admits a degree-1 operator Δ with $\Delta^2 = 0$, coming from the S^1-action on LM that rotates the free loop parameterization. Furthermore, the interaction between Δ and $\circ$ makes $\mathbb{H}_*(LM)$ a Batalin-Vilkovisky (BV) algebra, or, equivalently, an algebra over the homology of the framed little discs operad. Consequently, it is also a Gerstenhaber algebra, an algebra over the homology of the (unframed) little discs operad, via the loop product $\circ$ and the *loop bracket* $\{-,-\}$, a degree-1 Lie bracket defined in terms of $\circ$ and Δ.

Such algebraic structures arise in other mathematical contexts. For example, if A is a differential graded algebra, its Hochschild homology $HH_*(A)$ has a degree-1 Connes operator B with $B^2 = 0$, and its Hochschild cohomology $HH^*(A)$ is a Gerstenhaber algebra under the Hochschild cup product $\cup$ and the Gerstenhaber Lie bracket $[-,-]$. Consequently, it is natural to ask whether these constructions recover some of the structure of string topology for a choice of algebra A related to M. Two algebras that arise immediately as candidates are C^*M, the differential graded algebra of cochains of M under cup product, and $C_*\Omega M$, the algebra of chains on the based loop space ΩM of M, with product induced by the concatenation of based loops.

In the mid-1980s, Goodwillie and Burghelea and Fiedorowicz independently developed the first result of this form [3, 18], showing an isomorphism between $H_*(LX)$ and $HH_*(C_*\Omega X)$ for a connected space X that takes Δ to the B operator. Shortly after this result, Jones used a cosimplicial model for LM to show an isomorphism between $H^*(LX)$ and $HH_*(C^*X)$ when X is simply connected, taking a cohomological version of the Δ operator to B [22].

With the introduction of string topology, similar isomorphisms relating the loop homology $\mathbb{H}_*(LM)$ of M to the Hochschild cohomologies $HH^*(C^*M)$ and $HH^*(C_*\Omega M)$ were developed. One such family of isomorphisms arises from variations on the Jones isomorphism, and so also requires M to be simply connected. More closely reflecting the Burghelea-Fiedorowicz–Goodwillie perspective, Abbaspour, Cohen, and Gruher [1] instead show that, if M is a $K(G,1)$ manifold for G a discrete group, then there is an isomorphism of graded algebras between $\mathbb{H}_*(LM)$ and $H^*(G, kG^c)$, the group cohomology of G with coefficients in the group ring kG with the conjugation action. Vaintrob [46] notes that this is also isomorphic to $HH^*(kG)$ and shows that, when k is a field of characteristic 0, $HH^*(kG)$ admits a BV structure isomorphic to that of string topology.

Our main result is a generalization of this family of results, replacing the group ring kG with the chain algebra $C_*\Omega M$. When $M = K(G,1)$, $\Omega M \simeq G$, so $C_*\Omega M$ and $kG = C_*G$ are equivalent algebras.

Theorem 1.1.1 Let k be a commutative ring, and let X be a k-oriented, connected Poincaré duality space of dimension d. Poincaré duality, extended to allow $C_*\Omega X$-modules as local coefficients, gives a sequence of weak equivalences inducing an isomorphism of graded

k-modules

$$D : HH^*(C_*\Omega X) \to HH_{*+d}(C_*\Omega X).$$

Pulling back $-B$ along D gives a degree-1 operator $-D^{-1}BD$ on $HH^*(C_*\Omega X)$. This operator interacts with the Hochschild cup product to make $HH^*(C_*\Omega X)$ a BV algebra, where the induced bracket coincides with the usual bracket on Hochschild cohomology.

When X is a manifold as above, the composite of D with the Goodwillie isomorphism $HH_*(C_*\Omega X) \to H_*(LX)$ gives an isomorphism $HH^*(C_*\Omega X) \cong \mathbb{H}_*(LX)$ taking this BV algebra structure to that of string topology. ∎

We produce the D isomorphism in Theorem 4.1.1, and we establish the BV algebra structure on $HH^*(C_*\Omega M)$ and its relation to the string topology BV algebra in Theorems 5.3.7, and 5.3.8. Since the D isomorphism ultimately comes from Poincaré duality with local coefficients, this result also allows us to see more directly that the Chas-Sullivan loop product comes from the intersection product on the homology of M with coefficients taken in $C_*\Omega M$ with the loop-conjugation action.

We now indicate the structure of the rest of this dissertation. In Chapter 2, we provide background and preliminary material for our comparison of string topology and Hochschild homology. We state the basic properties of the singular chains C_*X of a space X, including the algebra structure when X is a topological monoid. We also develop the notions of Ext, Tor, and Hochschild homology and cohomology over a differential graded algebra A in terms of a model category structure on the category of A-modules, and we use two-sided bar constructions as models for this homological algebra. Via Rothenberg-Steenrod constructions, we relate these algebraic constructions to the topological setting. Additionally, we state the key properties of the loop product $\circ$ and BV operator Δ in string topology, and we survey previous connections between the homology of loop spaces and Hochschild homology and cohomology.

In Chapter 3, we develop the extended or "derived" Poincaré duality we use above, which originates in work of Klein [25] and of Dwyer, Greenlees, and Iyengar [9]. In this setting, we broaden the notion of local coefficient module for M to include modules over the DGA $C_*\Omega M$, instead of simply modules over $\pi_1 M$, and we show that Poincaré duality for $\pi_1 M$-modules implies Poincaré duality for this wider class of coefficients.

In Chapter 4, we relate the Hochschild homology and cohomology of $C_*\Omega M$ to this extended notion of homology and cohomology with local coefficients, where the coefficient module is $C_*\Omega M$ itself with an action coming from loop conjugation. In fact, there are several different models of this adjoint action that are convenient to use in different contexts, and in order to switch between them we must employ some technical machinery involving morphisms of A_∞-modules between modules over an ordinary DGA. In any case, this result combines with Poincaré duality to establish the isomorphism D above, coming from a sequence of weak equivalences on the level of chain complexes.

Chapter 5 relates the BV structure of the string topology of M to the algebraic structures present on the Hochschild homology and cohomology of $C_*\Omega M$. In order to do so, we must engage with a spectrum-level, homotopy-theoretic description of the Chas-Sullivan loop product. We show that the Thom spectrum LM^{-TM} and the topological Hochschild cohomology of $S[\Omega M]$, the suspension spectrum of ΩM, are equivalent as ring spectra, using techniques in fiberwise spectra from Cohen and Klein [8]. We recover the chain-level equivalences established earlier by smashing with the Eilenberg-Mac Lane spectrum Hk and passing back to the equivalent derived category of chain complexes over k.

We then show that the pullback $-D^{-1}BD$ of the B operator to $HH^*(C_*\Omega M)$ forms a BV algebra structure on $HH^*(C_*\Omega M)$, and that this structure coincides with that of string topology. We do this by establishing that D is in fact given by a Hochschild cap product against a fundamental class $z \in HH_d(C_*\Omega M)$, for which $B(z) = 0$. These two conditions allow us to apply an algebraic argument of Ginzburg [16], with some sign corrections by Menichi [34], to establish this BV algebra structure.

Appendix A contains various algebraic definitions, including our conventions regarding chain complexes, differential graded algebras, coalgebras, and Hopf algebras, and (bi)modules over DGAs. It also defines Gerstenhaber and Batalin-Vilkovisky algebras and includes a statement of the cofibrantly generated model category structure of the category of unbounded chain complexes. Finally, it contains an overview of the definitions of A_∞ algebras and A_∞ modules and how they relate to two-sided bar constructions and Ext and Tor.

Chapter 2

Background and Preliminaries

2.1 Singular Chain Complexes

2.1.1 Simplicial Sets and Properties of C_*X

We note our conventions regarding simplicial objects and singular complexes of topological spaces.

Definition 2.1.1 Let Δ denote the *simplicial category*, with objects

$$[n] = \{0 < 1 < \ldots < n\}$$

for $n \in \mathbb{N}$ and morphisms $\Delta([m],[n])$ all order-preserving maps from $[m]$ to $[n]$. If C is a category, then a *simplicial object* $F_\bullet$ in C is a functor $F : \Delta^{\mathrm{op}} \to C$, and a *cosimplicial object* $G^\bullet$ in C is a functor $G : \Delta \to C$.

Define the *geometric n-simplex* Δ^n by

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \right\},$$

with the subspace topology from $\mathbb{R}^{n+1}$. The elements of $[n]$ can be identified with the vertices of Δ^n, with i corresponding to the vertex $(0, \ldots, 1, \ldots, 0)$ with 1 in coordinate $i+1$. Extending $\phi \in \Delta([m],[n])$ linearly gives a map $\phi_* : \Delta^m \to \Delta^n$; the assignment $[n] \mapsto \Delta^n, \phi \mapsto \phi_*$

determines a cosimplicial space $\Delta^\bullet$. Let d^i and s^i denote its coface and codegeneracy maps.■

Recall that k denotes our fixed commutative ring. Let X be a topological space, and let F_k denote the free functor from Set to k-Mod.

Definition 2.1.2 Define the *singular complex* functor $S_\bullet$ by $S_n(X) = \mathsf{Top}(\Delta^n, X)$. Since $\Delta^\bullet$ is a cosimplicial space, $S_\bullet(X)$ is a simplicial set. Denote its face and degeneracy maps by d_i and s_i.

Let $CS_\bullet(X;k)$ be the simplicial k-module $F_k(S_\bullet(X))$, and let the (unnormalized) *chain complex* $CS_*(X;k)$ of X be the associated Moore chain complex, with differential $d = \sum_{i=0}^n (-1)^i d_i$. Let $DS_*(X;k)$ be the degenerate chain complex of X, defined levelwise by $DS_n(X;k) = \sum_i s_i(CS_*(X);k)$. Then define the *normalized chain complex* $C_*(X;k)$ of X to be the quotient $CS_*(X;k)/DS_*(X;k)$. ■

It is standard [17, §III.2] that the projection map $CS_*(X;k) \to C_*(X;k)$ is a chain homotopy equivalence, and thus that $H_*(C_*(X;k)) \cong H_*(CS_*(X;k)) = H_*(X;k)$. Also, we follow Schwede and Shipley's convention [37, §2.1] of taking the normalization functor N in the Dold-Kan correspondence to be exactly the quotient complex $NA = CA/DA$, rather than the usual subcomplex of A. Then $C_*(X;k) = N(C_\bullet(X;k))$. If X is a one-point space, then the only non-degenerate simplex is the unique map $\Delta^0 \to X$, so $C_*(X;k) \cong k$. Finally, the k is often dropped when the ground ring is understood from context, as are the parentheses when no ambiguities can arise.

2.1.2 Tensor Products and Eilenberg-Zilber Equivalences

Let Y be another topological space. We recall the standard natural Eilenberg-Zilber equivalences between $C_*X \otimes C_*Y$ and $C_*(X \times Y)$. Take an n-simplex $\phi \in S_n(X \times Y)$. Letting π_X and π_Y denote the projection maps, the n-simplices $\pi_X\phi : \Delta^n \to X$ and $\pi_Y\phi : \Delta^n \to Y$ uniquely determine ϕ by $\phi = (\pi_X\phi, \pi_Y\phi)$.

Definition 2.1.3 Define the *Alexander-Whitney map* $AW : C_*(X \times Y) \to C_*X \otimes C_*Y$ by

$$AW(\phi) = \sum_{i=0}^n d_{n-i+1} \cdots d_{n-1} d_n \pi_X \phi \otimes d_0^{n-i} \pi_Y \phi.$$

Similarly, define the *Eilenberg-Zilber map* $EZ : C_*X \otimes C_*Y \to C_*(X \times Y)$ on simplices $\rho : \Delta^n \to X$ and $\sigma : \Delta^m \to Y$ by

$$EZ(\rho \otimes \sigma) = \sum_{(\mu,\nu)\in S_{n,m}} (-1)^{\epsilon(\mu,\nu)} (s_{\nu(n)} \cdots s_{\nu(1)}\rho, s_{\mu(m)} \cdots s_{\mu(1)}\sigma),$$

where $S_{n,m}$ is the set of (n, m)-shuffles in S_{n+m}, and $\epsilon : S_{n+m} \to \{\pm 1\}$ is the sign homomorphism. Note that the $\mu(i)$ and $\nu(j)$ values together range from 0 through $n + m - 1$. ∎

Below are several standard facts about EZ and AW (see [11, p. 55]).

Proposition 2.1.4 Take spaces X', Y' and continuous maps $f : X \to X'$, $g : Y \to Y'$. Then

(*a*) AW and EZ are both natural, so the diagrams below commute:

$$\begin{array}{ccc} C_*X \otimes C_*Y & \xrightarrow{EZ} & C_*(X \times Y) \\ \downarrow{\scriptstyle C_*f \otimes C_*g} & & \downarrow{\scriptstyle C_*(f\times g)} \\ C_*X' \otimes C_*Y' & \xrightarrow{EZ} & C_*(X' \times Y') \end{array} \qquad \begin{array}{ccc} C_*(X \times Y) & \xrightarrow{AW} & C_*X \otimes C_*Y \\ \downarrow{\scriptstyle C_*(f\times g)} & & \downarrow{\scriptstyle C_*f \otimes C_*g} \\ C_*(X' \times Y') & \xrightarrow{AW} & C_*X' \otimes C_*Y' \end{array}$$

(*b*) AW and EZ are associative, so $(AW \otimes \mathrm{id})AW = (\mathrm{id} \otimes AW)AW$ and $EZ(EZ \otimes \mathrm{id}) = EZ(\mathrm{id} \otimes EZ)$.

(*c*) AW and EZ are compatible: let τ denote the map interchanging tensor factors defined in Section A.1.4, and let $t = t_{X,Y}$ denote the topological interchange map $t : X \times Y \to Y \times X$ given by $t(x, y) = (y, x)$, with the same notational conventions as for the τ morphisms. Then

$$\begin{array}{ccc} C_*(X \times X') \otimes C_*(Y \times Y') & \xrightarrow{AW \otimes AW} & C_*X \otimes C_*X' \otimes C_*Y \otimes C_*Y' \\ \downarrow{\scriptstyle EZ} & & \downarrow{\scriptstyle \mathrm{id} \otimes \tau \otimes \mathrm{id}} \\ C_*(X \times X' \times Y \times Y') & & C_*X \otimes C_*Y \otimes C_*X' \otimes C_*Y' \\ \downarrow{\scriptstyle C_*(\mathrm{id} \times t \times \mathrm{id})} & & \downarrow{\scriptstyle EZ \otimes EZ} \\ C_*(X \times Y \times X' \times Y') & \xrightarrow[AW]{} & C_*(X \times Y) \otimes C_*(X' \times Y') \end{array}$$

(*d*) EZ is compatible with interchange of factors: $C_*t \circ EZ = EZ \circ \tau$.

(*e*) $AW \circ EZ = \mathrm{id}$, and there exists a natural map H with $dH + Hd = EZ \circ AW - \mathrm{id}$, so that EZ and AW are chain homotopy inverses.

The equality $AW \circ EZ = \mathrm{id}$ holds only on the normalized chain complexes. ∎

Notation 2.1.5 Denote by $EZ_{X_1,\dots,X_n}$ and $AW_{X_1,\dots,X_n}$ the unique maps between $C_*X_1 \otimes \cdots \otimes C_*X_n$ and $C_*(X_1 \times \cdots \times X_n)$ determined by iterated EZ and AW maps, respectively. ▪

It is standard that for a space X with diagonal δ, $\Delta = AW \circ C_*\delta$ and $\epsilon : C_*X \to C_*(\mathrm{pt})$ make C_*X a counital differential graded coalgebra.

2.1.3 The Based Loop Space, Topological Monoids, and Group Models

Suppose now that X has a fixed basepoint x_0. Recall that the based loop space ΩX of X is homotopy equivalent to the Moore loop space MX, which has a strictly associative and unital multiplication by concatenation. Hence, MX is a topological monoid.

Let M be a topological monoid. Following Burghelea and Fiedorowicz [3, p. 311], define a simplicial group $G_\bullet = B(F, J, S_\bullet(M))$, where J is the James free monoid construction, F is the free group construction, and $B(-,-,-)$ is May's two-sided categorical bar construction [30]. Then the maps $|B(J, J, S_\bullet(M))| \to |S_\bullet(M)|$, $|S_\bullet(M)| \to M$, and $|B(J, J, S_\bullet(M))| \to |B(F, J, S_\bullet(M))|$ are all both homotopy equivalences and maps of monoids, so the zigzag

$$|G_\bullet| = |B(F, J, S_\bullet(M))| \leftarrow |B(J, J, S_\bullet(M))| \to |S_\bullet(M)| \to M$$

yields a simplicial topological group $G_\bullet$ homotopy equivalent to M. This construction applied to MX yields a group model G for ΩX. Another such construction is the Kan loop group $\tilde{G}_\bullet(K)$ of a simplicial set K [17, 23]. Then $|\tilde{G}_\bullet(S_\bullet X)|$ provides a topological group model for ΩX.

2.1.4 Properties of C_*G

Let G be a topological monoid, with identity element $e \in G$ and multiplication map $m : G \times G \to G$. Let $\iota : \{e\} \to G$ be the inclusion. It is standard that $\mu = C_*m \circ EZ$ and $\eta = C_*\iota$ make C_*G a differential graded algebra. Furthermore, μ and η are compatible with the coalgebra structure on C_*G, so C_*G is a differential graded Hopf algebra.

Suppose now that G is a topological group, with inverse map $i : G \to G$. We discuss the existence of an antipode map for the DGH C_*G.

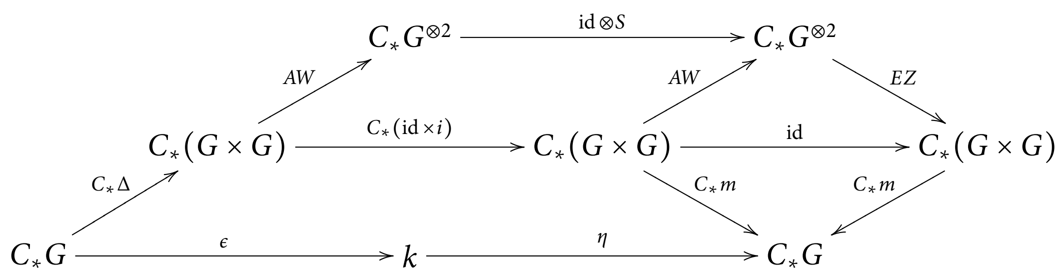
Proposition 2.1.6 Let $S = C_* i$. Then S is an algebra anti-automorphism of $C_* G$, $S^2 = \mathrm{id}$, and $\Delta(\mathrm{id} \otimes S)\mu \simeq \eta\epsilon$ (similarly for $S \otimes \mathrm{id}$).

Proof: We first show that $S = C_* i$ is a graded anti-automorphism of $C_* G$, so that $S \circ \mu = \mu \circ \tau \circ (S \otimes S)$. Since $i \circ m = m \circ t \circ (i \times i)$ and $C_* t \circ EZ = EZ \circ \tau$,

$$\begin{aligned} S \circ \mu = C_* i \circ C_* m \circ EZ = C_*(i \circ m) \circ EZ = C_*(m \circ t \circ (i \times i)) \circ EZ \\ = C_* m \circ EZ \circ \tau \circ (C_* i \otimes C_* i) = \mu \circ \tau \circ (S \otimes S). \end{aligned}$$

Since $i^2 = \mathrm{id}$, $(C_* i)^2 = \mathrm{id}$, so $C_* i$ is an involution of $C_* G$.

Finally, the antipode diagram for $1 \otimes S$ commutes up to chain homotopy, using the chain homotopy H from $EZ \circ AW$ to id:



■

A similar diagram holds for $S \otimes \mathrm{id}$.

2.2 Differential Graded Homological Algebra

We have seen that $C_* G$ provides a DGA for G a topological monoid, and we are now interested in developing suitable notions of homological algebra for modules over these DGAs. In order to develop constructions of homological algebra for modules over a DGA A, we determine a cofibrantly generated model category structure on the category A-Mod of A-modules. (Appendix A contains our conventions regarding DGAs, their modules, and cofibrantly generated model categories.)

This model category structure leads to suitable definitions of $\mathrm{Ext}^*_A(-,-)$ and $\mathrm{Tor}^A_*(-,-)$ as the homology of derived functors associated to $\mathrm{Hom}_A(-,-)$ and $- \otimes_A -$, respectively.

Furthermore, the cofibrantly generated model structure incorporates the notions of semifree extensions and resolutions of A-modules. In cases where the underlying algebra and modules are cofibrant as chain complexes of k-modules, two-sided bar constructions give convenient models for cofibrant replacement and hence for the derived functors of $\otimes_A$ and Hom_A. Additionally, these bar constructions arise in work of Félix, Halperin, and Thomas [10] that generalize the Rothenberg-Steenrod spectral sequence on the level of chain complexes, and we use them to connect these algebraic models to their topological applications.

2.2.1 A Model Category Structure on A-Mod

Recall from Section A.2 the cofibrantly generated model category structure on the category $\mathrm{Ch}(k)$ of unbounded chain complexes, with the cofibrations generated by the set I of maps $i_n : S^{n-1} \to D^n$ and the trivial cofibrations generated by the set J of maps $j_n : 0 \to D^n$.

Now suppose that A is a DGA over k. Let $F_A : \mathrm{Ch}(k) \to \mathrm{Ch}(k)$ denote the free A-module functor $A \otimes -$. F_A is a monad in $\mathrm{Ch}(k)$, with the natural transformations $F_A F_A \to F_A$ and $I \to F_A$ arising from the multiplication and unit maps of A. Furthermore, the category of left A-modules is precisely the category of algebras of the monad F_A. Let $I_A = F_A(I)$ and $J_A = F_A(J)$ be the images of the sets I under F_A. As a consequence of [36, Lemma 2.3], the category A-Mod admits a model category structure if A is cofibrant:

Proposition 2.2.1 Suppose that A is a cofibrant object in $\mathrm{Ch}(k)$. Then A-Mod has a cofibrantly generated model category structure with I_A as the set of generating cofibrations and J_A as the set of generating trivial cofibrations. A morphism in A-Mod is a weak equivalence or a fibration if the underlying morphism of chain complexes is one.

Proof: We verify the hypotheses of Lemma 2.3 of [36]; the stated characterization of the weak equivalences and fibrations is part of the conclusion of the lemma. Since F_A is given by tensor product with A, it commutes with filtered colimits. Since chain complexes are small, the domains of I_A and J_A are small relative to I_A-cell and J_A-cell.

Finally, since A is cofibrant, F_A preserves trivial cofibrations by the pushout product axiom, and so each element of J_A is a trivial cofibration in $\mathrm{Ch}(k)$. Thus, the morphisms in J_A-cell are all trivial cofibrations in $\mathrm{Ch}(k)$, and hence are all weak equivalences in A-Mod. As the morphisms in J_A-cell are the regular J_A-cofibrations in the terminology of [36], condition (1) in the lemma is satisfied, and so A-Mod admits the desired cofibrantly generated model category structure. ∎

Homological algebra for A-modules has also been discussed in terms of semifree extensions and resolutions. We review the definitions of these notions and relate them to the model category structure exhibited above.

Definition 2.2.2 An A-module P is a *semifree extension* of M if P is a union of an increasing family of A-submodules $P(-1) \subset P(0) \subset \cdots$ such that $P(-1) = M$ and each $P(k)/P(k-1)$ is A-free on a basis of cycles. If $M = 0$, we say P is an *A-semifree module*.

Let $f : M \to N$ be a morphism of A-modules. A *semifree resolution* of f is a semifree extension P of M with a quasi-isomorphism $P \xrightarrow{\simeq} N$ extending f.

A *semifree resolution* of an A-module N is a semifree resolution of $0 \to N$. ∎

Proposition 2.2.3 The class of semifree extensions coincides with I_A-cell.

Proof: Suppose P is a semifree extension of M. We show each $P(n-1) \hookrightarrow P(n)$ is a map of I_A-cell. By the definition of a semifree extension, as graded k-modules $P(n) \cong P(n-1) \oplus (A \otimes V(n))$, where $V(n)$ is free on a basis $\{v_j\}_{j\in J}$ with each $dv_j \in P(n-1)$. Hence, the following is a pushout diagram:

$$\begin{array}{ccc} \bigoplus_{j\in J} A \otimes S^{|v_j|-1} & \xrightarrow{\oplus dv_j} & P(n-1) \\ \downarrow{\scriptstyle \oplus F_A(i_{|v_j|})} & & \downarrow \\ \bigoplus_{j\in J} A \otimes D^{|v_j|} & \xrightarrow{\oplus v_j} & P(n) \end{array}$$

Consequently, $P(n-1) \to P(n)$ is in I_A-cell. Since P is the colimit of these maps over n, $M \to P$ is also in I_A-cell.

Conversely, suppose $f : M \to P$ is a morphism in I_A-cell. Then f is a transfinite composition of pushouts along the $F_A(i_n)$ morphisms. Note that a morphism $A \otimes S^n \to M'$ of A-modules is determined solely by the image of $1 \otimes 1$ in M'. The pushout P_β after any stage in the transfinite composition is isomorphic as a graded k-module to a direct sum of M and copies of A, and so the image of $1 \otimes 1$ in P_β lies in only a finite number of factors of this direct sum. Using this finiteness, the transfinite composition can be reorganized into a countable sequence of pushout diagrams as above, thus exhibiting $M \to P$ as a semifree extension. ∎

As a result of the characterization of cofibrations in a cofibrantly generated model category, we obtain the following connections between cofibrations of A-modules and semifree extensions.

Corollary 2.2.4 If $i : M \to P$ is a semifree extension of A-modules, then it is a cofibration. A map $i : M \to N$ is a cofibration if and only if i is a retract of a semifree extension $j : M \to P$. An A-module Q is cofibrant if and only if it is a retract of a semifree A-module F, i.e., an A-module direct summand of F. ∎

Furthermore, several useful results from [10, §2] regarding semifree resolutions generalize to statements about cofibrations of A-modules. The most general one is as follows, where ϕ^* denotes the pullback notation of Appendix A.1.4.

Proposition 2.2.5 Suppose that

(a) $\phi : B \to A$ is a DGA morphism,

(b) P is a cofibrant B-module, Q a cofibrant A-module, and $f : P \to \phi^* Q$ a morphism of B-modules,

(c) $g : \phi^* M \to N$ is a morphism of left A-modules,

(d) $h : S \to \phi^* T$ is a morphism of right B-modules.

Then if ϕ, f, g, h are all quasi-isomorphisms, so are $h \otimes_\phi f : S \otimes_B P \to T \otimes_A Q$ and $\mathrm{Hom}_\phi(f, g) : \mathrm{Hom}_A(Q, M) \to \mathrm{Hom}_B(P, N)$. ∎

Taking $B = A$ above yields the following useful results:

Proposition 2.2.6 Suppose P and Q are cofibrant (left or right, as is appropriate) A-modules.

(a) $P \otimes_A -$ and $\mathrm{Hom}_A(P, -)$ are exact functors from A-Mod to $\mathrm{Ch}(k)$ and preserve all weak equivalences in A-Mod.

(b) If $f : P \to Q$ is a weak equivalence in A-Mod, then $f \otimes_A M$ and $\mathrm{Hom}_A(f, M)$ are also weak equivalences for all A-modules M.

Proof: The only statements requiring verification are those concerning exactness. Since P is cofibrant, it is a retract of a colimit of free A-modules, so these exactness properties follow.

2.2.2 Defining Ext_A^* and Tor_*^A

Since $- \otimes_A -$ and $\mathrm{Hom}_A(-,-)$ preserve sufficiently large classes of weak equivalences, they admit total left and right derived functors, and hence the category of A-modules admits analogues of the classical Ext and Tor derived functors. We review these notions briefly below. We follow Hovey [21] in requiring that the factorization axioms for a model category produce *functorial* factorizations. Consequently, each model category admits cofibrant and fibrant replacement functors denoted Q and R, respectively. By this functoriality, Q comes with a natural trivial fibration $q_M : QM \to M$, and, dually, R comes with a natural trivial cofibration $r_M : M \to RM$.

Definition 2.2.7 Suppose M and N are left A-modules and P is a right A-module. Define the total left derived functor $\otimes_A^L$ of $\otimes_A$ to be $P \otimes_A^L M = QP \otimes_A QM$, where Q is the cofibrant replacement functor above. Similarly, the total right derived functor $R\,\mathrm{Hom}_A$ of Hom_A is defined to be $R\,\mathrm{Hom}_A(M,N) = \mathrm{Hom}_A(QM, RN)$.

Define $\mathrm{Tor}_n^A(P,M) = H_n(P \otimes_A^L M)$ and $\mathrm{Ext}_A^n(M,N) = H_n(R\,\mathrm{Hom}_A(M,N))$. ∎

Note that these derived functors take values in the homotopy category of $\mathrm{Ch}(k)$, where weak equivalences are inverted. By Proposition 2.2.6, the derived functors are up to isomorphism independent of the choice of cofibrant replacement, and so the Ext and Tor modules defined above are also independent of this choice.

Since all A-modules are fibrant, the existence of the cofibrant replacement functor Q implies that any two weakly equivalent cofibrant objects M, N of A-Mod are homotopy equivalent.

Proposition 2.2.8 If M, N are cofibrant A-modules connected by a zigzag of weak equivalences, there is a homotopy equivalence $h : M \to N$ homotopic to this zigzag in $\mathrm{Ho}\,A$-Mod.

Proof: We first observe that if $f : P \to P'$ is a weak equivalence between cofibrant A-modules P, P', then f is a homotopy equivalence.

Apply the cofibrant replacement functor Q to the zigzag of weak equivalences connecting M and N. Then this is a zigzag of weak equivalences between cofibrant A-modules, hence a zigzag of weak equivalences. By choosing homotopy inverses where needed, there is a homotopy equivalence $g : QM \to QN$ in the same homotopy class as this zigzag. Let $h = q_N g q_M^{-1}$, where q_M^{-1} is a homotopy inverse to the weak, hence homotopy, equivalence $q_M : QM \to M$. ∎

2.2.3 Bar constructions, Ext, and Tor

Several constructions of Ext and Tor over a DGA A exist in the literature [3, 12, 18, 22, 28]. We describe them in terms of bar constructions over A. Such constructions are general enough to be performed in any monoidal category.

Definition 2.2.9 Let $(\mathsf{C}, \otimes, I)$ be a monoidal category and let (A, μ, η) be a monoid in C. Given a right A-module M and a left A-module N, define the *(two-sided) bar construction* to be the simplicial C-object $B_\bullet(M, A, N)$, with $B_n(M, A, N) = M \otimes A^{\otimes n} \otimes N$, $k \geq 0$. The face and degeneracy maps d_i and s_i are given by

$$s_i = \mathrm{id}^{\otimes i+1} \otimes \eta \otimes \mathrm{id}^{\otimes n+1-i}, \qquad d_i = \begin{cases} a_M \otimes \mathrm{id}^n, & i = 0, \\ \mathrm{id}^{\otimes i} \otimes \mu \otimes \mathrm{id}^{\otimes n-i}, & 1 \leq i \leq n-1, \\ \mathrm{id}^{\otimes n} \otimes a_N, & i = n. \end{cases}$$ ∎

In the case where $\mathsf{C} = \mathrm{Ch}(k)$ and A is therefore a DGA, write $m[a_1 \mid \cdots \mid a_k]n$ for the element $m \otimes a_1 \otimes \cdots \otimes a_k \otimes n$. The differential d in $B_k(M, A, N)$ is given by the graded tensor product of the differentials of the factors; explicitly, this is

$$\begin{aligned} d(m[a_1 \mid \cdots \mid a_k]n) = dm[a_1 \mid \cdots \mid a_k]n + \sum_{i=1}^{k} (-1)^{|m|+|a_1|+\cdots+|a_{i-1}|} m[a_1 \mid \cdots \mid da_i \mid \cdots \mid a_k]n \\ + (-1)^{|m|+|a_1|+\cdots+|a_k|} m[a_1 \mid \cdots \mid a_k]dn. \end{aligned}$$

Definition 2.2.10 Let $X_\bullet$ be a simplicial chain complex, and let $\Delta^\bullet$ denote the standard cosimplicial simplicial set of simplices, with $\Delta^n_m = \Delta([m], [n])$. Then $k\Delta^\bullet$ is the associated cosimplicial simplicial k-module, and $N(k\Delta^\bullet)$ is the cosimplicial chain complex obtained by applying the Dold-Kan normalization functor N levelwise. Following the perspective of [39,

§12], define the *geometric realization* of $X_\bullet$ to be the coend $|X_\bullet| = N(k\Delta^\bullet) \otimes_{\Delta^{\mathrm{op}}} X_\bullet$. Likewise, define the *thick realization* of $X_\bullet$ to be $\|X_\bullet\| = k\Delta^\bullet \otimes_{\Delta^{\mathrm{op}}} X_\bullet$.

Define $B(M,A,N) = \|B_\bullet(M,A,N)\|$ and $\bar{B}(M,A,N) = |B_\bullet(M,A,N)|$. ■

As discussed above, we define $N(k\Delta^n)$ to be the quotient complex $k\Delta^n/D(k\Delta^n)$, taking the nondegenerate simplices of $k\Delta(m,n)$ as a basis for $N(k\Delta^n)_m$ and taking the differential to be the Moore complex differential, $d_s = \sum_{i=0}^k (-1)^i d_i$. This differential makes $B_\bullet(M,A,N)$ a chain complex of chain complexes $B_*(M,A,N)$. Examining the enriched coend of Definition 2.2.10 shows that $B(M,A,N) \cong \mathrm{Tot}(B_*(M,A,N))$. The differential on the factor $B_p(M,A,N)$ is $d_s + (-1)^p d$.

Furthermore, for M (resp., N) an A-A-bimodule, $B(M,A,N)$ is a left (resp., right) A-module. In particular, then, $B(A,A,M)$ is a left A-module. In this case, there is a map of A-modules $q_M : B(A,A,M) \to M$ given by $q_M(a[\,]m) = am$ and $q_M(a[a_1 \mid \cdots \mid a_k]m) = 0$. This map q_M is a fibration and a weak equivalence of A-modules.

Corollary 2.2.11 If M is semifree as a chain complex of k-modules, then $B(A,A,M)$ is a semifree A-module, and so is an explicit cofibrant replacement for M. The Ext and Tor modules then admit explicit expressions as

$$\begin{aligned} \mathrm{Ext}_A^*(M,N) &\cong H_*(\mathrm{Hom}_A(B(A,A,M),N)), \\ \mathrm{Tor}_*^A(P,M) &\cong H_*(B(P,A,A) \otimes_A M) = H_*(B(P,A,M)). \end{aligned}$$ ■

Consequently, under the appropriate cofibrancy conditions, these bar constructions provide a combinatorial construction of the complexes representing Ext_A^* and Tor_*^A.

2.2.4 Hochschild Homology and Cohomology

With these combinatorial models for Ext and Tor in mind, we define a homology and cohomology theory for bimodules over a DGA A. See Appendix A.1.4 for our conventions regarding bimodules and the canonical $A^e = A \otimes A^{\mathrm{op}}$ module structures on such bimodules.

Definition 2.2.12 Given a DGA A and an A-A-bimodule M, define the *Hochschild homology* of A with coefficients in M to be $HH_*(A,M) = \mathrm{Tor}_*^{A^e}(M,A)$, treating M canonically as a right A^e-module and A as a left A^e-module. Similarly, define the *Hochschild cohomology* of A with coefficients in M to be $HH^*(A, M) = \mathrm{Ext}^*_{A^e}(A, M)$, where M is now canonically a left A^e-module.

When $M = A$, considered as a bimodule over itself, we write $HH_*(A)$ for $HH_*(A, A)$ and $HH^*(A)$ for $HH^*(A, A)$. ▪

In the case when $M = N = A$, $B(A, A, A)$ is an A-A-bimodule, and hence canonically a left A^e-module. Since A is assumed to be cofibrant in $\mathrm{Ch}(k)$, $B(A, A, A)$ is a cofibrant A^e-module, weakly equivalent to A.

Definition 2.2.13 The *Hochschild (co)chains* of A with coefficients in M are

$$CH_*(A, M) = M \otimes_{A^e} B(A, A, A) \quad \text{and} \quad CH^*(A, M) = \mathrm{Hom}_{A^e}(B(A, A, A), M).$$ ▪

Then Hochschild homology and cohomology over the DGA A may be expressed as the homology of these Hochschild chains and cochains. Since $M \otimes_{A^e} B_n(A, A, A)$ and $M \otimes A^{\otimes n}$ are canonically isomorphic as chain complexes for all $n \geq 0$, the simplicial structure on the former induces one on the latter, which yields the definition of $HH_*(A, M)$ given in the literature. Furthermore, in the case where A and M are concentrated in degree 0, these definitions reduce to the usual simplicial definitions of Hochschild homology and cohomology on an ungraded unital k-algebra.

These combinatorial descriptions of the Hochschild chains and cochains give rise to additional operations on Hochschild homology and cohomology when $M = A$. First, the Hochschild homology $HH_*(A)$ of A admits a degree-1 operator B with $B^2 = 0$ due to Connes [28], arising from the cyclic permutation of the $n + 1$ A factors in the nth level of $CH_*(A, A)$.

The operations on Hochschild cohomology are most easily described on the homogeneous pieces $\mathrm{Hom}_{A^e}(A^{\otimes n+2}, A)$ of $CH^*(A, A)$, which are isomorphic to $\mathrm{Hom}_k(A^{\otimes n}, A)$. The cup product $f \cup g$ of cochains $f : A^{\otimes p} \to A$ and $g : A^{\otimes q} \to A$ is the map $\mu(f \otimes g) : A^{\otimes p+q} \to A$ given by applying f to the first p A tensor factors and g to the remaining q tensor factors, and then multiplying the two A output tensor factors. This cup product operation respects cocycles and coboundaries, and hence defines a cup product $\cup$ on $HH^*(A)$. Moreover, the cup product is homotopy commutative on cochains, and so gives a graded-commutative product on $HH^*(A)$. The cup product can also be described on the derived level via the Yoneda or composition product on $R\operatorname{Hom}_{A^e}(A, A)$.

Gerstenhaber also identifies a degree-1 Lie bracket $[-,-]$ on $HH^*(A)$ arising from composition of cochains [14]. Given cochains f, g as above, let $f \circ_i g$ denote the composite of f and g where the output of g is the ith input of f. Then the bracket $[f, g]$ is commutator-like expression

$$[f, g] = \sum_{i=1}^{p} (-1)^{i(|g|-1)} f \circ_i g - (-1)^{|f||g|} \sum_{i=1}^{q} (-1)^{i(|f|-1)} g \circ_i f.$$

Stasheff gives an alternate description of the bracket by extending f and g to coderivations on the coalgebra $B(k, A, k)$; their bracket is then the commutator $[f, g]$ of the coderivations. In any case, the cup product and the bracket interact to make $HH^*(A)$ a Gerstenhaber algebra.

Additionally, $HH_*(A, M)$ is a right module for the algebra $HH^*(A)$. This module structure can be seen both combinatorially and on the derived level. In the latter context, we observe that $R\operatorname{Hom}_{A^e}(A, A)$ acts on the A factor in $M \otimes^L_{A^e} A$. Passing to homology, and recalling from above that the composition product on $R\operatorname{Hom}_{A^e}(A, A)$ induces the cup product in $HH^*(A)$, this action induces a cap product $HH_*(A, M) \otimes HH^*(A) \to HH_*(A, M)$ making $HH_*(A, M)$ a right module for $HH^*(A)$.

Combinatorially, for a chain $z \in CH_{p+q}(A, M) \cong M \otimes A^{p+q}$ and a cochain $f \in CH^p(A)$, evaluation of f on the first p A factors of z gives a chain $z \cap f$, and this map descends to homology to give the same right action of $HH^*(A)$ on $HH_*(A, M)$. When $M = A$, this cap product is part of a calculus structure on $(HH^*(A), HH_*(A))$ that formalizes the interaction of differential forms and (poly)vector fields on a manifold [43].

2.2.5 Rothenberg-Steenrod constructions

In the case where $A = C_*G$ for G a topological monoid, Félix et al. [10] determine several results which generalize the Rothenberg-Steenrod spectral sequence [31, §7.4][35] to equivalences of chain complexes and of differential graded coalgebras, phrased in terms of bar constructions.

Definition 2.2.14 ([10]) A G-fibration consists of a surjective fibration $\pi : E \to X$ and a continuous right action $\mu_E : E \times G \to E$ such that, for all $x \in X$, $E_x \cdot G \subset E_x$, and for all $z \in Z$, the map $a \mapsto za$ is a weak homotopy equivalence from G to $E_{\pi(z)}$. ▪

In particular, the Moore path space fibration $PX \to X$, with $G = \Omega X$, is a G-fibration. The principal results of interest in our case are as follows [10, Thm 5.1, Thm 6.3, Prop. 6.13]:

Theorem 2.2.15 Suppose that $\pi : E \to X$ is a G-fibration. Then there is a natural quasi-isomorphism of differential graded coalgebras $B(C_*E, C_*G, k) \xrightarrow{\simeq} C_*X$. ▪

Theorem 2.2.16 For any path connected space X, the DGC C_*X is weakly DGC-equivalent to the bar construction $B(k, C_*\Omega X, k)$. ▪

Proposition 2.2.17 Let G be a topological group and let F be a right G-space. Then the DGC $C_*(F \times_G EG)$ is weakly DGC-equivalent to $B(C_*F, C_*G, k)$. ▪

2.3 String Topology and Hochschild Constructions

2.3.1 String Topology Operations

We describe some of the conventions and fundamental operations in string topology. Let M be a closed, smooth, k-oriented manifold of dimension d, and let $LM = \mathrm{Map}(S^1, M)$ be the space of free loops in M, taking $S^1 = \mathbb{R}/\mathbb{Z} = \Delta^1/\partial\Delta^1$ as our model for S^1.

For M any space, note that S^1 acts on LM, with the action map $\rho : S^1 \times LM \to LM$ given by $\rho(t,\gamma)(s) = \gamma(s+t)$. Then ρ induces a map

$$H_p(S^1) \otimes H_q(LM) \xrightarrow{\times} H_{p+q}(S^1 \times LM) \xrightarrow{\rho_*} H_{p+q}(LM).$$

For $\alpha \in H_p(LM)$, define $\Delta(\alpha) = \rho_*([S^1] \times \alpha)$, where $[S^1] \in H_1(S^1)$ is the fundamental class of S^1 determined by the quotient map $\Delta^1 \to \Delta^1/\partial\Delta^1$. Then Δ is a degree-1 operator on $H_*(LM)$. Since degree considerations force $\mu_*([S^1] \times [S^1]) \in H_2(S_1)$ to be 0, Δ^2 is identically 0.

With a different choice $[S^1]'$ for the fundamental class, so that $[S^1]' = \lambda[S^1]$ for $\lambda \in k^\times$, then the corresponding operator Δ' is $\lambda\Delta$. In particular, choosing the opposite orientation for the cycle $\Delta^1 \to \Delta^1/\partial\Delta^1$, $t \mapsto 1-t$, yields the operator $-\Delta$.

We postpone detailed discussion of the Chas-Sullivan loop product on $H_*(LM)$ until Section 5.2.1, where we give a homotopy-theoretic construction using Thom spectra due to

Cohen and Jones [7]. For now, we record that the loop product arises from a combination of the degree-$(-d)$ intersection product on $H_*(M)$ and of the Pontryagin product on $H_*(\Omega M)$ induced by concatenation of based loops. Consequently, the loop product also exhibits a degree shift of $-d$:

$$\circ : H_p(LM) \otimes H_q(LM) \to H_{p+q-d}(LM).$$

In order that $\circ$ define a graded algebra structure, we shift $H_*(LM)$ accordingly:

Definition 2.3.1 Denote $\Sigma^{-d}H_*(LM)$ as $\mathbb{H}_*(LM)$, called the *loop homology* of M, so that $\mathbb{H}_q(LM) = H_{q+d}(LM)$. ▪

Under this degree shift, Δ gives a degree-1 operator on $\mathbb{H}_*(LM)$. The key result of Chas and Sullivan is that $\circ$ and Δ interact to give a BV algebra structure on $\mathbb{H}_*(LM)$. As discussed in Section A.1.6, this BV algebra structure gives a canonical Gerstenhaber algebra structure, and the resulting Lie bracket, denoted $\{-,-\}$, is called the *loop bracket*. The loop bracket can also be defined more directly using operations on Thom spectra [6, 42].

2.3.2 Relations to Hochschild Constructions

There are already substantial connections known between the homology and cohomology of the free loop space LX of a space X and the Hochschild homology and cohomology of the DGAs $C_*\Omega X$ and C^*X. We state the key results that we employ below and survey the other relevant results.

The main result we will use is due to Goodwillie [18, §V] and Burghelea and Fiedorowicz [3, Theorem A].

Theorem 2.3.2 For X a connected space, there is an isomorphism $BFG : HH_*(C_*\Omega X) \to H_*LM$ of graded k-modules, such that $BFG \circ B = \Delta \circ BFG$. ▪

The proofs of this statement essentially rely on modeling the free loop space LX as a cyclic bar construction on ΩX, or a topological group replacement. Dually, Jones has shown that, for X a simply connected space, $HH_*(C^*X) \cong H^*(LX)$, taking B to a cohomological version of the Δ operator [22]. Jones's construction uses a cosimplicial model for LX, coming from the cyclic cobar construction on the space X itself.

In their homotopy-theoretic construction of string topology, Cohen and Jones modify this cyclic cobar construction to produce a cosimplicial model for the Thom spectrum LM^{-TM} in terms of the manifold M and the Atiyah dual M^{-TM} of M [7]. Applying chains and Poincaré duality to this cosimplicial model yields an isomorphism

$$\mathbb{H}_*(LM) \cong HH^*(C^*M)$$

of graded algebras, taking the loop product to the Hochschild cup product. As with Jones's earlier result, this isomorphism requires M to be simply connected. When k is a field of with char $k = 0$, Félix and Thomas have identified a BV-algebra structure on $HH^*(C^*M)$ and have shown that it coincides with the string topology BV structure under this isomorphism [13].

Koszul duality also provides a class of results relating the Hochschild cohomologies of different DGAs and hence providing other characterizations of string topology. In particular, Félix, Menichi, and Thomas have shown that, for C a simply connected DGC with $H_*(C)$ of finite type, there is an isomorphism $HH^*(C^*) \cong HH^*(\Omega C)$ of Gerstenhaber algebras, where C^* is the k-linear dual of C and where ΩC is the cobar algebra of C [12]. When $C = C_*M$ for a simply connected manifold M, $C^* \cong C^*M$ and $\Omega C \simeq C_*\Omega M$, so $HH^*(C^*M) \cong HH^*(C_*\Omega M)$ as Gerstenhaber algebras. Combining this result with the isomorphism of Cohen and Jones gives an isomorphism of graded algebras $HH^*(C_*\Omega M) \cong \mathbb{H}_*(LM)$, again in the simply connected case.

Proceeding more directly from the $C_*\Omega M$ perspective above, Abbaspour, Cohen, and Gruher have characterized the string topology of an aspherical d-manifold $M = K(G,1)$ in terms of the group homology of the discrete group G [1]. In particular, in this setting G is a Poincaré duality group, and they established a multiplication on the shifted group homology $H_{*+d}(G, kG^c)$, coming from a G-equivariant convolution product on $H^*(G; kG^c)$. They also established an isomorphism $\mathbb{H}_*(LM) \cong H_{*+d}(G, kG^c)$ of graded algebras. By classical Hopf-algebra arguments, these group homology and cohomology groups are isomorphic to the Hochschild homology and cohomology of the group algebra kG. When k is a field with char $k = 0$, Vaintrob has shown that $HH^*(kG)$ has a BV algebra structure and that this isomorphism is one of BV algebras [46].

Consequently, our main result Theorem 1.1.1 can be viewed as a generalization of these results to the case where M is an arbitrary connected manifold and where k is a general commutative ring for which M is oriented.

Chapter 3

Derived Poincare Duality

3.1 Derived Local Coefficients

Now that we have appropriate constructions for $\mathrm{Ext}^*_A(M,N)$ and $\mathrm{Tor}^A_*(M,N)$, as well as explicit models for the chain complexes $R\,\mathrm{Hom}_A(M,N)$ and $M\otimes^L_A N$ for A, M, N k-cofibrant, we establish a duality isomorphism between Ext and Tor when $A = C_*\Omega M$. As we explain below, these results are analogous to related duality results of Klein [26] for topological groups and of Dwyer, Greenlees, and Iyengar [9] on connective ring spectra satisfying a form of Poincaré duality.

We first generalize the notion of a local coefficient system on X. Suppose that X is connected. Then by the Rothenberg-Steenrod constructions above,

$$H_*(X) \cong H_*(B(k, C_*\Omega X, k)) = \mathrm{Tor}^{C_*\Omega X}_*(k,k).$$

Moreover, the Borel construction $E(\Omega X)\times_{\Omega X}\pi_1 X$ provides a model for the universal cover $\tilde{X}$ of X, with the right action by $\pi_1 X$. Consequently,

$$B(k, C_*\Omega X, k[\pi_1 X])$$

provides a model for the right $k[\pi_1 X]$-module $C_*(\tilde{X};k)$. Analogously, $B(k[\pi_1 X], C_*\Omega X, k)$ models $C_*(\tilde{X};k)$ with the left $\pi_1 X$-action.

Suppose that E is a system of local coefficients in the usual sense, i.e., a right $k[\pi_1 X]$-module. Under the morphism $C_*\Omega X \to H_0(\Omega X) = k[\pi_1 X]$ of DGAs, E is a $C_*\Omega X$-module, and

$$\begin{aligned} C_*(X;E) &= E \otimes_{k[\pi_1 X]} C_*(\tilde{X}) \\ &\simeq E \otimes_{k[\pi_1 X]} B(k[\pi_1 X], C_*\Omega X, k) \cong B(E, C_*\Omega X, k) \simeq E \otimes^L_{C_*\Omega X} k. \end{aligned}$$

Similarly,

$$\begin{aligned} C^*(X;E) &= \mathrm{Hom}_{k[\pi_1 X]}(C_*(\tilde{X}), E) \\ &\simeq \mathrm{Hom}_{C_*\Omega X}(B(k, C_*\Omega X, C_*\Omega X), E) \simeq R\,\mathrm{Hom}_{C_*\Omega X}(k, E). \end{aligned}$$

Passing to homology,

$$H_*(X;E) \cong \mathrm{Tor}_*^{C_*\Omega X}(E,k) \quad \text{and} \quad H^*(X;E) \cong \mathrm{Ext}^*_{C_*\Omega X}(k,E).$$

Hence, $E \otimes^L_{C_*\Omega X} k$ and $R\,\mathrm{Hom}_{C_*\Omega X}(k,E)$ provide a generalization of homology and cohomology with local coefficients, where the coefficients are now $C_*\Omega X$-modules and where these theories take values in the derived category $\mathrm{Ho}\, Ch(k)$ of chain complexes over k.

Definition 3.1.1 For a $C_*\Omega X$-module E, let $H_\bullet(X;E) = E \otimes^L_{C_*\Omega X} k$, and let $H^\bullet(X;E) = R\,\mathrm{Hom}_{C_*\Omega X}(k,E)$. Let $H_*(X;E)$ and $H^*(X;E)$ denote their homologies. ▪

When X is a Poincaré duality space, however, these "derived" versions of homology and cohomology satisfy a "derived" version of Poincaré duality:

Theorem 3.1.2 Suppose X is a k-oriented Poincaré duality space of dimension d. Let $z \in \mathrm{Tor}_d^{C_*\Omega X}(k,k)$ correspond to the fundamental class $[X] \in H_d(X)$. For E a right $C_*\Omega M$-module, there is an evaluation map

$$\mathrm{ev}_{z,E} : H^\bullet(X;E) \to \Sigma^{-d} H_\bullet(X;E)$$

that is a weak equivalence. On homology, this produces an isomorphism

$$H^*(X;E) \to H_{*+d}(X;E).$$

When E is a $k[\pi_1 X]$-module considered as a module over $C_*\Omega X$, this isomorphism coincides with the isomorphism coming from Poincaré duality for X with local coefficients E. ∎

We relate these results to analogous ones in other algebraic and topological contexts. In group cohomology, it is well-known [2] that, for a discrete Poincaré duality group G of dimension d and a kG-module M, there exist isomorphisms $\mathrm{Ext}^i_{kG}(k,M) \cong \mathrm{Tor}^{kG}_{d-i}(k,M)$ for all $i \geq 0$, induced by cap product with a distinguished class $z \in \mathrm{Tor}^{kG}_d(k,k)$.

Our approach to establishing this duality for $C_*\Omega X$ has been heavily influenced by Klein's results for topological groups [26], which we summarize here. If G is a topological group such that BG is a finitely dominated G-complex of formal dimension d, then the G-spectrum

$$D_G = S[G]^{hG} = F(EG_+, S[G])^G$$

is weakly equivalent to S^{-d}, and has a right G-action from the remaining action on $S[G]$. Consequently, for E a (naive) G-spectrum, there is a norm map $D_G \wedge_{hG} E \to E^{hG}$, and under the hypotheses on G, it is a weak equivalence of spectra. Considering $D_G \wedge_{hG} -$ and $-^{hG}$ as the appropriate derived functors of $D_G \wedge_G -$ and $-^G = F(S^0,-)^G$ in the category of G-spectra, these results are a spectrum-level generalization of the classical Poincaré-duality results for discrete groups.

The arguments that establish this duality result for G-spectra rely on the notion of an equivariant duality map [25]. In the category of based G-spaces, this is a map $d : S^n \to Y \wedge_G Z$ for Y, Z cofibrant and homotopy finite such that for all G-spectra E, the map taking $f : \Sigma^j Y \to E_{j+k}$ to $(f \wedge_G Z) \circ \Sigma^j d : S^{n+j} \to E_{j+k} \wedge_G Z$ induces isomorphisms $E^*_G(Y) \to E^G_{*-n}(Z)$ of cohomology groups.

Furthermore, Klein establishes that such equivariant duality maps can be detected from a simpler criterion. Let $\pi = \pi_0(G)$ and let $G_0 \subset G$ be the path-component of the identity, so G_0 is the kernel of the projection map $G \to \pi$. To check whether $S^n \to Y \wedge_G Z$ is a G-duality map, it suffices to check whether the composite $S^n \to Y \wedge_G Z \to Y_{G_0} \wedge_\pi Z_{G_0}$ is a π-equivariant duality map with respect to $H\mathbb{Z}\pi$, the Eilenberg-Mac Lane π-spectrum of the integral group ring $\mathbb{Z}\pi$.

We establish analogous characterizations of finiteness for A-modules, and we exhibit a similar detection result when A is a chain DGA. Dwyer, Greenlees, and Iyengar [9, §10.4] also proceed essentially following these ideas of Klein to establish similar results for modules over the ring spectrum $\Sigma^\infty \Omega X_+ \wedge k$, where k is a commutative ring spectrum.

3.2 Duality for A-Modules

We review the notions of finiteness and duality in the category of A-modules coming from the cofibrantly generated model category structure.

Definition 3.2.1 Recall the generating cofibrations $A \otimes i_n : A \otimes S^{n-1} \to A \otimes D^n$ from Section 2.2. An A-module M is *finite free* if there exists a finite sequence $M_0, \dots, M_n$ of A-modules such that $M_0 = 0$, $M_n = M$, and for each $j = 1, \dots, n$ there exists an n_j such that

$$\begin{array}{ccc} A \otimes S^{n_j - 1} & \longrightarrow & M_{j-1} \\ {\scriptstyle A \otimes i_{n_j}} \downarrow & & \downarrow \\ A \otimes D^{n_j} & \longrightarrow & M_j \end{array}$$

is a pushout square, so that M is built from 0 by a finite number of pushouts along the $A \otimes i_{n_j}$. M is *finite* if it is a retract (i.e., a direct summand) of a finite free A-module P, and M is *homotopy finite* if there exists a zigzag of weak equivalences between M and a finite A-module. ▪

Since a finite free A-module is constructed from 0 from a finite sequence of pushouts along cofibrations $A \otimes i_{n_j}$, it is also cofibrant. By the closure of cofibrations under retracts, finite A-modules are also all cofibrant.

Definition 3.2.2 Suppose that P and M are right A-modules and Q is a left A-module. An element $z \in (P \otimes_A Q)_n$ defines a degree-n linear map $\mathrm{ev}_{z,M} : \mathrm{Hom}_A(P, M) \to M \otimes_A Q$ by $\mathrm{ev}_{z,M}(f) = (f \otimes \mathrm{id}_Q)(z)$. If z is a cycle, then $\mathrm{ev}_{z,M}$ is a cycle as well. Thus, if z is a 0-cycle, $\mathrm{ev}_{z,M}$ is a chain map of complexes.

If instead M is a left A-module, we define $\mathrm{ev}_{z,M} : \mathrm{Hom}_A(Q, M) \to P \otimes_A M$ by $\mathrm{ev}_{z,M}(f) = (\mathrm{id}_P \otimes f)(z)$.

Passing to derived constructions, a class $\alpha \in H_0(P \otimes^L_A Q)$ induces a map

$$\mathrm{ev}_{\alpha,M} : R\,\mathrm{Hom}_A(P, M) \to M \otimes^L_A Q$$

well defined up to homotopy (and hence well-defined in the homotopy category). Such a class α is a *dualizing class with respect to M* if $\mathrm{ev}_{\alpha,M}$ is a weak equivalence, and α is a *dualizing class* if it is one with respect to all (left and right) A-modules. ∎

By definition, then, a dualizing class $\alpha \in H_0(P \otimes^L_A Q)$ induces isomorphisms

$$\mathrm{Ext}^*_A(P, M) \to \mathrm{Tor}^A_*(M, Q)$$

for all A-modules M.

Finite A-modules satisfy a form of strong duality that is a generalization of the duality for finitely generated projective modules over an (ordinary) ring (cf. Brown [2, §1.8]). As noted in Section A.1.3, for $M \in A\text{-Mod}$, $M^* = \mathrm{Hom}_A(M, A)$ is in $\text{Mod-}A$, with the right A-module structure given explicitly by $(fa)(m) = (-1)^{|m||a|} f(m)a$. Similarly, if $M \in \text{Mod-}A$ is a right module, $M^* \in A\text{-Mod}$, with $(af)(m) = af(m)$.

Proposition 3.2.3 Suppose P is a finite right A-module.

(a) P^* is a finite left A-module.

(b) Let N be a right A-module. Then the map $\phi_N : N \otimes_A P^* \to \mathrm{Hom}_A(P, N)$, given by $\phi(n \otimes f)(p) = nf(p)$, is an isomorphism.

(c) Let N be a left A-module. Then the map $\phi'_N : P \otimes_A N \to \mathrm{Hom}_A(P^*, N)$, given by $\phi'(p \otimes n)(f) = (-1)^{|f|(|p|+|n|)} f(p)n$ for homogeneous $f \in P^*$, $n \in N$, and $p \in P$, is an isomorphism.

(d) The map $\phi'' : P \to (P^*)^*$ of right A-modules given by $\phi''(u)(x) = u(x)$ is an isomorphism.

Proof: For P finite free with A-generators $x_1, \dots, x_n$, P^* is finite free on generators given by the duals x_i^* of the x_i, with x_n^* attached first, then x_{n-1}^*, and so on down to x_1^* in reverse order.

Evaluation against the Casimir element $z = \sum_i x_i \otimes x_i^*$ in $P \otimes_A P^*$ gives an explicit inverse to ϕ and ϕ', and ϕ'' is ϕ' when $N = A$. Since these maps split over finite direct sums, the isomorphisms hold when P is a summand of a finite free module. ∎

These arguments also show that if P is a homotopy finite A-module, there is a (finite) A-module Q and a dualizing class $\alpha \in H_0(P \otimes_A^L Q)$, evaluation against which induces weak equivalences $R\operatorname{Hom}_A(P, M) \simeq M \otimes_A^L Q$ for all A-modules M.

Following Klein, we now wish to know a more basic criterion to determine whether a given class α as above is a dualizing class.

Notation 3.2.4 For A a chain DGA, let $\tilde{A} = H_0 A$, and let $\pi : A \to \tilde{A}$ be the surjective map taking $a \in A_0$ to the class $[a]$, and taking $a \in A_n$ to 0 for $n > 0$.

For a left (resp., right) A-module M, let $\tilde{M}$ be the $\tilde{A}$-module $\pi^* \tilde{A} \otimes_A M$ (resp., $M \otimes_A \pi^* \tilde{A}$). Let $\pi_M : M \to \pi^* \tilde{M}$ be the surjective map of A-modules given by $\pi_M(m) = [1] \otimes m$. ∎

Suppose that M is a left A-module. Then a computation shows that the graded k-module $H_* M$ admits a left action by the graded ring $H_* A$, with $[a] \cdot [m] = [am]$ for classes $[a] \in H_*(A)$ and $[m] \in H_* M$. In particular, then, each $H_j M$ is a left $\tilde{A}$-module.

Theorem 3.2.5 Suppose M, N are cofibrant, homotopy finite A-modules. Take $z \in H_0(M \otimes_A N)$, and let $z_0 = (\pi_M \otimes_\pi \pi_N)_*(z) \in H_0(\tilde{M} \otimes_{\tilde{A}} \tilde{N})$. Then z is a dualizing class if z_0 is a dualizing class for the $\tilde{A}$-module $\tilde{A}$.

Proof: It suffices to reduce to the case when M, N are finite. Since M, N are cofibrant and homotopy finite, there exist finite A-modules F and G with homotopy equivalences $f : M \to F$ and $g : N \to G$. Let $w = (f \otimes g)(z)$. Note that $\tilde{F}$ and $\tilde{G}$ are finite $\tilde{A}$-modules, since $\otimes$ commutes with colimits, and that f and g induce homotopy equivalences $\tilde{f} : \tilde{M} \to \tilde{F}$ and $\tilde{g} : \tilde{N} \to \tilde{G}$ of $\tilde{A}$-modules, with $\tilde{f}\pi_M = \pi_F f$ and $\tilde{g}\pi_N = \pi_G g$. Hence, $w_0 = (\pi_F \otimes_\pi \pi_G)(w) = (\tilde{f} \otimes \tilde{g})(z_0)$. Consequently, the diagrams

$$\begin{CD}
\operatorname{Hom}_{\tilde{A}}(\tilde{F}, \tilde{A}) @>{\mathrm{ev}_{w_0,\tilde{A}}}>> \tilde{A} \otimes_{\tilde{A}} \tilde{G} \\
@V{\operatorname{Hom}_{\tilde{A}}(\tilde{f},\tilde{A})}V{\simeq}V @A{\simeq}A{\mathrm{id} \otimes \tilde{g}}A \\
\operatorname{Hom}_{\tilde{A}}(\tilde{M}, \tilde{A}) @>{\mathrm{ev}_{z_0,\tilde{A}}}>> \tilde{A} \otimes_{\tilde{A}} \tilde{N}
\end{CD}
\qquad
\begin{CD}
\operatorname{Hom}_A(F, E) @>{\mathrm{ev}_{w,E}}>> E \otimes_A G \\
@V{\operatorname{Hom}_A(f,E)}V{\simeq}V @A{\simeq}A{\mathrm{id} \otimes g}A \\
\operatorname{Hom}_A(M, E) @>{\mathrm{ev}_{z,E}}>> E \otimes_A N
\end{CD}$$

commute. Therefore, if $\mathrm{ev}_{z_0,A}$ is a weak equivalence, so is $\mathrm{ev}_{w_0,A}$, and if $\mathrm{ev}_{w,E}$ is a weak equivalence, so is $\mathrm{ev}_{z,E}$.

Consequently, suppose throughout that M, N are finite. Let E be a right $\tilde{A}$-module, and consider the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_A(M, E^*\pi) & \xrightarrow{\mathrm{ev}_{z,\pi^* E}} & \pi^* E \otimes_A N \\
\downarrow \cong & & \\
\mathrm{Hom}_A(M, \mathrm{Hom}_{\tilde{A}}(\pi^*\tilde{A}, E)) & & \uparrow \cong \\
\downarrow \cong & & \\
\mathrm{Hom}_{\tilde{A}}(M \otimes_A \pi^*\tilde{A}, E) & \xrightarrow{\mathrm{ev}_{z_0,E}} & E \otimes_{\tilde{A}} (\pi^*\tilde{A} \otimes_A N) \\
\phi'_E \uparrow \cong & & \| \\
E \otimes_{\tilde{A}} \mathrm{Hom}_{\tilde{A}}(\tilde{M}, \tilde{A}) & \xrightarrow[\simeq]{E \otimes \mathrm{ev}_{z_0,\tilde{A}}} & E \otimes_{\tilde{A}} \tilde{N}
\end{array}$$

where the left-hand vertical maps are isomorphisms from $\pi^* E \cong \mathrm{Hom}_{\tilde{A}}(\pi^*\tilde{A}, E)$ and from the adjoint associativity isomorphism, and where the right-hand vertical isomorphisms are obvious. A computation shows the top and bottom rectangles commute. Since $\tilde{M}$ is finite, $\tilde{M}^* = \mathrm{Hom}_{\tilde{A}}(\tilde{M}, \tilde{A})$ is finite as well, and so the weak equivalence $\mathrm{ev}_{z_0,\tilde{A}} : \tilde{M}^* \to \tilde{N}$ is a homotopy equivalence. Hence, $E \otimes \mathrm{ev}_{z_0,\tilde{A}}$ is a weak equivalence, so therefore $\mathrm{ev}_{z_0,E}$ and then $\mathrm{ev}_{z,\pi^* E}$ are weak equivalences as well.

Now let E be an A-module. For each $j \in \mathbb{Z}$, define the A-submodule $K_j E$ of E by

$$(K_j E)_i = \begin{cases} E_i, & i > j+1, \\ \ker d \subset E_{j+1}, & i = j+1, \\ 0, & i \le j, \end{cases}$$

with the differential induced from that on E. Since A is non-negatively graded, $\ker d \subset E_{j+1}$ is an A_0-module, and so $K_j E$ is in fact an A-module. Furthermore, by construction, $H_i(K_j E) = H_i(E)$ for $i > j$ and is 0 for $i \le j$.

Define the A-module $P_j E$ to be $E/K_j E$, with projection map $\pi_j : E \to P_j E$. (Note that $K_j E$ and $P_j E$ are essentially the good truncations [47, §1.2.7] of E.) By direct computation, or by the long exact sequence in homology, $H_i(P_jE) = H_iE$ for $i \geq j$ and is 0 for $i > j$. Furthermore, the K_jE give a sequence of increasing submodules of A, and the inclusions $K_jE \hookrightarrow K_{j-1}E$ induce surjections $p_j : P_jE \to P_{j-1}E$.

Define $F_jE = \ker p_j$, so that $(F_jE)_{j+1} = E_{j+1}/\ker d$ and $(F_jE)_j = \ker d \subset E_j$. Consequently, $H_*(F_jE) \cong \Sigma^j H_jE$, and in fact the projection map $F_jE \to \Sigma^j H_jE$ is a weak equivalence of A-modules. Hence ev_{z,F_jE} is a weak equivalence. Additionally, since $0 \to F_jE \to P_jE \to P_{j-1}E \to 0$ is exact, the exactness of $\mathrm{Hom}_A(M,-)$ and $-\otimes_A N$ and the naturality of ev_z yield a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(M, F_jE) & \longrightarrow & \mathrm{Hom}_A(M, P_jE) & \longrightarrow & \mathrm{Hom}_A(M, P_{j-1}E) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}} & & \\ 0 & \longrightarrow & F_jE \otimes_A N & \longrightarrow & P_jE \otimes_A N & \longrightarrow & P_{j-1}E \otimes_A N & \longrightarrow & 0 \end{array}$$

and therefore a long exact sequence in homology:

$$\begin{array}{ccccccccc} \mathrm{Ext}_A^{i+1}(M, P_{j-1}E) & \to & \mathrm{Ext}_A^{i}(M, F_jE) & \to & \mathrm{Ext}_A^{i}(M, P_jE) & \to & \mathrm{Ext}_A^{i}(M, P_{j-1}E) & \to & \mathrm{Ext}_A^{i-1}(M, F_jE) \\ \downarrow{\scriptstyle \mathrm{ev}_*} & & \downarrow{\scriptstyle \mathrm{ev}_*} & & \downarrow{\scriptstyle \mathrm{ev}_*} & & \downarrow{\scriptstyle \mathrm{ev}_*} & & \downarrow{\scriptstyle \mathrm{ev}_*} \\ \mathrm{Tor}^A_{i+1}(P_{j-1}E, N) & \to & \mathrm{Tor}^A_{i}(F_jE, N) & \to & \mathrm{Tor}^A_{i}(P_jE, N) & \to & \mathrm{Tor}^A_{i}(P_{j-1}E, N) & \to & \mathrm{Tor}^A_{i-1}(F_jE, N) \end{array}$$

Now let E be a bounded-below A-module, so that $E = \oplus_{i \geq N} E_i$ for some sufficiently small N. Then $K_{N-1}E = E$, so $P_{N-1}E = 0$, and thus $\mathrm{ev}_{z,P_{N-1}E}$ is a weak equivalence. Using the five lemma inductively on the map of long exact sequences above, each ev_{z,P_jE} is a weak equivalence.

Suppose that E' is a left A-module that is bounded below. Then for a fixed $i \in \mathbb{Z}$, the map

$$(\pi_j \otimes E')_i : (E \otimes_A E')_i \to (P_jE \otimes_A E')_i$$

is an isomorphism for all j sufficiently large. Consequently, for such j, $H_i(\pi_j \otimes_A E')$ is an isomorphism.

Similarly, suppose that E' is a left A-module with a set $\{x_\gamma\}_{\gamma \in \Gamma}$ of A-generators that is bounded above in degree by L. Then for a given i, and for $f \in \mathrm{Hom}_A(E', E)_i$, the $f(x_\gamma)$ lie in degree no higher than $i + L$. Since the $f(x_\gamma)$ determine f on all of E' by A-linearity, for $j \geq i + L$, $\mathrm{Hom}_A(E', E)_i \to \mathrm{Hom}_A(E', P_jE)_i$ is an isomorphism. Then for all j sufficiently large, $H_i(\mathrm{Hom}_A(E', \pi_j))$ is also an isomorphism.

If E' is finite, it is bounded below, and is finitely generated over A, so that the π_j yield the above isomorphisms in H_i for j sufficiently large. Since M and N are both assumed finite, $H_i(N \otimes \pi_j)$ and $H_i(\mathrm{Hom}_A(M, \pi_j))$ are isomorphisms for j sufficiently large. By the naturality of ev_z with respect to the coefficient module, these isomorphisms imply $H_i(\mathrm{ev}_{z,E})$ is an isomorphism for all i, and hence that $\mathrm{ev}_{z,E}$ is a weak equivalence.

Finally, let E be an arbitrary A-module. Then $E = \mathrm{colim}_{j \to -\infty} K_j E$, where each $K_j E$ is bounded below. Since $N \otimes_A -$ is a left adjoint, it commutes with this colimit, as does the computation of H_*, so $H_*(N \otimes_A E) \cong \mathrm{colim}_j H_*(N \otimes_A K_j E)$. Similarly, $\mathrm{Hom}_A(M, -)$ commutes with colimits since M is finite, so $H_*(\mathrm{Hom}_A(M, E)) \cong \mathrm{colim}_j H_*(\mathrm{Hom}_A(M, K_j E))$. Since $\mathrm{ev}_{z,K_j E}$ is an isomorphism for all j, the naturality of ev_z induces an isomorphism on the colimits, and $\mathrm{ev}_{z,E}$ is a weak equivalence for all E. ∎

3.3 Reinterpretation of Poincaré Duality

Suppose now that X is a finite CW-complex satisfying Poincaré duality of formal dimension d with respect to all $k[\pi_1 X]$-modules. Thus, for a given such module E, capping with a fundamental class $[X] \in H_d(X; k)$ induces isomorphisms

$$H^*(X; E) \to H_{*+d}(X; E)$$

where, as usual, we view cohomology as being nonpositively graded. We reinterpret these properties in the context of the duality statements presented in the previous section.

First, we show that k is homotopy finite as a $C_*\Omega X$ module. More generally, adapting results of Félix et al. [10, Prop. 5.3] and Dwyer, Greenlees, and Iyengar [9, Prop. 5.3] shows the following:

Proposition 3.3.1 Suppose that $p : E \to X$ is a G-fibration. Then there exists a cofibrant right C_*G-module M and a quasi-isomorphism $m : M \to C_*E$ such that $\overline{m} : M \otimes_{C_*G} k \to C_*X$ is also a quasi-isomorphism. ∎

Corollary 3.3.2 When X is a finite CW-complex, k is a homotopy finite $C_*\Omega X$-module.

Proof: Consider the path-loop ΩX-fibration $PX \to X$. Then $C_*PX \to k$ is a quasi-isomorphism of $C_*\Omega X$-modules, and the construction of Proposition 3.3.1 yields a finite, semifree $C_*\Omega X$-module M and a quasi-isomorphism $M \to C_*PX$. Hence, M and k are weakly equivalent. ∎

Corollary 3.3.3 When X is finite, $B_*(k, C_*\Omega X, C_*\Omega X)$ is a homotopy finite and cofibrant $C_*\Omega X$-module weakly equivalent to k. ∎

We now return to the proof of Theorem 3.1.2.

Proof (Theorem 3.1.2): Recall that for a right $k[\pi_1 X]$-module E, the homology and cohomology groups $H_*(X; E)$ and $H^*(X; E)$ with local coefficients E are defined to be the homologies of $E \otimes_{k[\pi_1 X]} C_*\tilde{X}$ and $\mathrm{Hom}_{k[\pi_1 X]}(C_*\tilde{X}, E)$, respectively.

Let $A = C_*\Omega X$, so $\tilde{A} = H_0(\Omega X) \cong k[\pi_1 X]$. Let

$$M = B(k, A, A) \quad \text{and} \quad N = B(A, A, k),$$

so that M and N are cofibrant, weakly equivalent to k, and thus homotopy finite (since k is). Since $X \simeq B(\Omega X)$, $E(\Omega X) \times_{\Omega X} \pi_1 X$ is a model for a universal cover $\tilde{X}$ of X with a right $\pi_1 X$-action. Consequently, $C_*\tilde{X}$ is weakly equivalent to $\tilde{M} = B(k, A, \tilde{A})$ by Prop. 2.2.17. Similarly, $C_*\tilde{X}$ with the left $\pi_1 X$-action is weakly equivalent to $\tilde{N} = B(\tilde{A}, A, k)$.

Observe that C_*X is weakly equivalent to $B(k, A, k) \simeq M \otimes_A N \simeq \tilde{M} \otimes_{\tilde{A}} \tilde{N}$. Let $[X] \in H_d(X)$ be a choice of fundamental class for X. Then let $z \in H_d(M \otimes_A N)$ be the class corresponding to $[X]$, and let $z_0 = (\pi_M \otimes_\pi \pi_N)_*(z) \in H_d(\tilde{M} \otimes_{\tilde{A}} \tilde{N})$. Note that z_0 also corresponds to $[X]$. Let E be an $\tilde{A}$-module. Then z_0 induces

$$\mathrm{ev}_{z_0,E} : \mathrm{Hom}_{\tilde{A}}(\tilde{M}, E) \to E \otimes_{\tilde{A}} \Sigma^{-d}\tilde{N}.$$

Since $\mathrm{ev}_{z_0,E}$ corresponds to cap product, this map is the Poincaré duality isomorphism for the local coefficient module E, so $\psi_{z_0,E}$ is a weak equivalence.

In particular, taking $E = \tilde{A}$, z_0 is a dualizing class for the module $\tilde{A}$, so applying Theorem 3.2.5, z is a dualizing class for all A-modules. In particular, $\mathrm{ev}_{z,E} : \mathrm{Hom}_A(M, E) \to$

$E \otimes_A \Sigma^{-d} N$ is a weak equivalence for all E, and it induces an isomorphism $\mathrm{Ext}^*_A(k, E) \to \mathrm{Tor}^A_{*+d}(E, k)$.

Rephrasing this in terms of "derived" homology and cohomology with local coefficients, cap product with $[X]$ induces a weak equivalence $\mathrm{ev}_{[X]} : H^\bullet(X; E) \to \Sigma^{-d} H_\bullet(X; E)$ and hence an isomorphism $H^*(X; E) \to H_{*+d}(X; E)$. ∎

Chapter 4

Hochschild Homology and Cohomology

4.1 Hochschild Homology and Poincaré Duality

Now that we have established a Poincaré duality isomorphism $H^*(X;E) \to H_{*+d}(X;E)$ for X a k-oriented Poincaré duality space of dimension d and E an arbitrary $C_*\Omega X$-module, we use it to construct an isomorphism between $HH^*(C_*\Omega X)$ and $HH_{*+d}(C_*\Omega X)$.

Theorem 4.1.1 For X as above, derived Poincaré duality and the Hopf-algebraic properties of $C_*\Omega X$ produce a sequence of weak equivalences

$$\begin{array}{ccc} CH^*(C_*\Omega X) & \xrightarrow{\simeq} & H^\bullet(X;\mathrm{Ad}(\Omega X)) \\ & & \Big\downarrow {\scriptstyle \simeq}\ \mathrm{ev}_{[X]} \\ CH_{*+d}(C_*\Omega X) & \xrightarrow{\simeq} & H_{\bullet+d}(X;\mathrm{Ad}(\Omega X)) \end{array}$$

where CH_* and CH^* denote the Hochschild chains and cochains and where $\mathrm{Ad}(\Omega X)$ is $C_*\Omega X$ as a module over itself by conjugation, to be defined more precisely in Definition 4.2.3. In homology, this gives an isomorphism $D : HH^*(C_*\Omega X) \to HH_{*+d}(C_*\Omega X)$ of graded k-modules. ∎

Corollary 4.1.2 The composite of D and the Goodwillie isomorphism produces an additive isomorphism $HH^*(C_*\Omega X) \cong H_{*+d}(LX)$. ∎

In order to prove this result, we produce the horizontal isomorphisms relating the Hochschild homology and cohomology of $C_*\Omega X$ to $\mathrm{Ext}^*_{C_*\Omega X}$ and $\mathrm{Tor}_*^{C_*\Omega X}$. In fact, these equivalences hold for the algebra C_*G of chains on a topological group G, and we develop them in that generality.

4.2 Applications to C_*G

When A is a Hopf algebra with strict antipode S, then the Hochschild homology and cohomology of A can be expressed in terms of $\mathrm{Ext}^*_A(k,-)$ and $\mathrm{Tor}^A_*(-,k)$, using the isomorphisms of Prop. A.1.18. In particular, let M be an A-A-bimodule, considered canonically as a right A^e-module. Then the Hochschild chains and cochains are isomorphic to

$$\begin{aligned} CH_*(A,M) &= M\otimes_{A^e} B(A,A,A) \cong M\otimes_{A^e} B(\mathrm{ad}_0^*\,A^e,A,k)\cong B(\mathrm{ad}_0^*\,M,A,k),\\ CH^*(A,M) &= \mathrm{Hom}_{A^e}(B(A,A,A),M)\cong \mathrm{Hom}_{A^e}(B(k,A,\mathrm{ad}_1^*\,A^e),M)\\ &\cong \mathrm{Hom}_A(B(k,A,A),\mathrm{Hom}_{A^e}(\mathrm{ad}_1^*\,A^e,M))\cong \mathrm{Hom}_A(B(k,A,A),\mathrm{ad}_1^*\,M),\end{aligned}$$

which represent $\mathrm{Tor}^A_*(\mathrm{ad}_0^*\,M,k)$ and $\mathrm{Ext}^*_A(k,\mathrm{ad}_1^*\,M)$. These isomorphisms generalize the classical isomorphisms [28, Ex. 1.1.4] $HH_*(kG,M)\cong H_*(G,\mathrm{ad}^*\,M)$ and $HH^*(kG,M)\cong H^*(G,\mathrm{ad}^*\,M)$ when A is the group ring of a discrete group G. (Since kG is cocommutative, $\mathrm{ad}_0=\mathrm{ad}_1$, so these pullbacks coincide.)

Suppose now that G is a topological group, $A=C_*G$, and $S=C_*i$. Recall from Proposition 2.1.6 that $S^2=\mathrm{id}$, that $S: A\to A^{\mathrm{op}}$ is a DGA isomorphism, and that S satisfies the antipode identity for the DGH C_*G only up to chain homotopy equivalence. Nevertheless, we show that we can relate the Hochschild homology and cohomology of C_*G to $\mathrm{Ext}^*_{C_*G}(k,-)$ and $\mathrm{Tor}_*^{C_*G}(-,k)$.

Recall from Section A.1.5 that, since C_*G is a DGH and S is an anti-automorphism of C_*G, $\mathrm{ad}_0=(1\otimes S)\Delta$ and $\mathrm{ad}_1=(1\otimes S)\tau\Delta$ give DGA morphisms from A to A^e. Hence, pulling back along ad gives an adjoint A-module structure to A-A-bimodules.

This adjoint A-module structure plays a key role in relating the Hochschild homology and cohomology of C_*G to its Ext and Tor groups. In particular, since it is a pullback of an A^e-module, it behaves well with respect to both the formation of tensor products and

Hom-complexes, and hence with respect to the Hom-⊗ adjunction. As an intermediate result towards Theorem 4.1.1, we will therefore establish the following homotopy equivalences of A^e-modules.

Theorem 4.2.1 $B(\mathrm{ad}_0^* C_*G^e, C_*G, k)$ and $B(C_*G, C_*G, C_*G)$ are homotopy equivalent as left C_*G^e-modules, and $B(k, C_*G, \mathrm{ad}_0^* C_*G^e)$ and $B(C_*G, C_*G, C_*G)$ are homotopy equivalent as right C_*G^e-modules. ∎

As a consequence of this theorem, we immediately obtain relations between Hochschild constructions and Ext and Tor over C_*G:

Corollary 4.2.2 For $M \in A\text{-}\mathsf{Mod}\text{-}A$, considered canonically as a right A^e-module, there are weak equivalences

$$\Lambda_\bullet(G, M) : M \otimes_{A^e} B(A, A, A) \xrightarrow{\simeq} B(\mathrm{ad}_0^* M, A, k),$$
$$\Lambda^\bullet(G, M) : \mathrm{Hom}_{A^e}(B(A, A, A), M) \xrightarrow{\simeq} \mathrm{Hom}_A(B(k, A, A), \mathrm{ad}_0^* M).$$

When G and M are understood, we omit them from the notation. Passing to homology, these induce isomorphisms

$$\Lambda_*(G, M) : HH_*(A, M) \to \mathrm{Tor}_*^A(\mathrm{ad}_0^* M, k) \quad \text{and}$$
$$\Lambda^*(G, M) : HH^*(A, M) \to \mathrm{Ext}_A^*(k, \mathrm{ad}_0^* M).$$

Proof: Since $\mathrm{ad}_0^* M \cong M \otimes_{A^e} \mathrm{ad}_0^* A^e$, the homotopy equivalence $B(\mathrm{ad}_0^* A^e, A, k) \to B(A, A, A)$ of Theorem 4.2.1 yield a weak equivalence $\Lambda_\bullet(G, M)$,

$$M \otimes_{A^e} B(A, A, A) \simeq M \otimes_{A^e} B(\mathrm{ad}_0^* A^e, A, k) \cong B(\mathrm{ad}_0^* M, A, k),$$

which induces the isomorphism $\Lambda_*(G, M) : HH_*(A, M) \to \mathrm{Tor}_*^A(\mathrm{ad}_0^* M, k)$. Likewise, since $\mathrm{ad}_0^* M \cong \mathrm{Hom}_{A^e}(\mathrm{ad}_0^* A^e, M)$, the homotopy equivalence $B(k, A, \mathrm{ad}_0^* A^e) \to B(A, A, A)$ of Theorem 4.2.1 yields a weak equivalence $\Lambda^\bullet(G, M)$,

$$\mathrm{Hom}_{A^e}(B(A, A, A), M) \simeq \mathrm{Hom}_{A^e}(B(k, A, \mathrm{ad}_0^* A^e), M) \cong \mathrm{Hom}_A(B(k, A, A), \mathrm{ad}_0^* M),$$

which induces the isomorphism $\Lambda^*(G, M) : HH^*(A, M) \to \operatorname{Ext}^*_A(k, \operatorname{ad}_0^* M)$.

When G and M are clear from context, we may drop them from the notation. ▪

The C_*G-module $\operatorname{ad}_0^* M$ is actually not quite the definition we have in mind for the statement of Theorem 4.1.1. We introduce this slightly more natural adjoint module:

Definition 4.2.3 Let G^{op} be the group G with the opposite multiplication. Suppose that X is a space with a left action a_X by $G \times G^{\mathrm{op}}$. Pullback along $(\mathrm{id} \times i)\delta : G \to G \times G^{\mathrm{op}}$ makes X a left G-space by "conjugation," with $(g, x) \mapsto gxg^{-1}$. Let $\operatorname{Ad}_L(X)$ be C_*X with the corresponding left C_*G-module structure.

Similarly, we may produce a right C_*G-module structure $\operatorname{Ad}_R(X)$ on C_*X. We denote these modules simply as $\operatorname{Ad}(X)$ when the module structure is clear from context. ▪

Note that these $\operatorname{Ad}(X)$ modules arise from first converting the $G \times G^{\mathrm{op}}$-action into a G-action and then applying C_*. Consequently, these modules arise more naturally in topological contexts, although they are not as immediately compatible with tensor product and Hom-complex constructions.

Let K be another group. Note that if X has a right K-action commuting with the left $G \times G^{\mathrm{op}}$-action, then $\operatorname{Ad}(X)$ is a C_*G-C_*K-bimodule.

With the application of standard simplical techniques, the introduction of these Ad modules immediately provides a key intermediate step towards Theorem 4.2.1. First, we recall the two-sided bar construction in the topological setting.

Definition 4.2.4 Let X be a left G-space and Y a right G-space. Then the two-sided bar construction $B_\bullet(X, G, Y)$ is a simplicial space. Let $B(Y, G, X)$ be its geometric realization $|B_\bullet(Y, G, X)|$. ▪

Proposition 4.2.5 The maps EZ induce $B(C_*Y, C_*G, C_*X) \to C_*(B(Y, G, X))$ a weak equivalence.

Proof: Take $n \geq 0$. Observe that $B_n(C_*Y, C_*G, C_*X) = C_*Y \otimes C_*G^{\otimes n} \otimes C_*X$, and that

$$EZ : C_*Y \otimes C_*G^{\otimes n} \otimes C_*X \to C_*(Y \times G^n \times X) = C_*(B_n(Y, G, X))$$

is a chain homotopy equivalence. Denote this map by EZ_n. By the form of the face and degeneracy maps d_i and s_i for $B_\bullet(Y,G,X)$ and for $B_\bullet(C_*Y, C_*G, C_*X)$, it follows that $C_*(d_i)EZ_n = EZ_{n-1}d_i$ and $C_*(s_i)EZ_n = EZ_{n+1}s_i$ for all n and $0 \leq i \leq n$. Hence, the EZ_* assemble to a chain homotopy equivalence

$$EZ : \operatorname{Tot} B_*(C_*Y, C_*G, C_*X) \to \operatorname{Tot} C_*(B_\bullet(Y,G,X)).$$

Finally, since there is a weak equivalence from $\operatorname{Tot} C_*(E_\bullet)$ to $C_*(|E_\bullet|)$ for any simplicial space $E_\bullet$, [18, §V.1]the composite gives the desired weak equivalence. ∎

Note also that if Y or X has an action by another group H commuting with that of G, then the weak equivalence above is one of C_*H-modules.

Corollary 4.2.6 In the notation above, taking $Y = *$ and $X = G \times G^{\text{op}}$ with the standard left and right $G \times G^{\text{op}}$ actions yields a weak equivalence of right $C_*(G \times G^{\text{op}})$-modules

$$B(k, C_*G, \operatorname{Ad}(G \times G^{\text{op}})) \to C_*(B(*, G, G \times G^{\text{op}}))$$

Taking $X = Y = G$ yields a weak equivalence of right $C_*(G)^e$-modules

$$B(C_*G, C_*G, C_*G) \to EZ^* C_*(B(G,G,G)).$$

∎

We now relate the two topological bar constructions $B(*, G, G \times G^{\text{op}})$ and $B(G,G,G)$.

Proposition 4.2.7 There are homeomorphisms of right $G \times G^{\text{op}}$-spaces $\phi^R : B(*, G, G \times G^{\text{op}}) \leftrightarrows B(G,G,G) : \gamma^R$ and of left $G \times G^{\text{op}}$-spaces $\phi^L : B(G \times G^{\text{op}}, G, *) \leftrightarrows B(G,G,G) : \gamma^L$.

Proof: As G is a Hopf-object with antipode in Top with its usual symmetric monoidal structure, these homeomorphisms come from the simplicial Hopf-object isomorphisms of Prop. A.1.18. In particular, they define simplicial maps $\phi^R_\bullet : B_\bullet(*, G, G \times G^{\text{op}}) \leftrightarrows B_\bullet(G,G,G) : \gamma^R_\bullet$ by

$$\phi^R_n(g_1, \dots, g_n, (g, g')) = (g'(g_1 \cdots g_n)^{-1}, g_1, \dots, g_n, g),$$
$$\gamma^R_n(g', g_1, \dots, g_n, g) = (g_1, \dots, g_n, (g, g' g_1 \cdots g_n))$$

Since $(g_1, \ldots, g_n, (g, g')) \cdot (h, h') = (g_1, \ldots, g_n, (gh, h'g'))$ and $(g', g_1, \ldots, g_n, g) \cdot (h, h') = (h'g', g_1, \ldots, g_n, gh)$, they are maps of right $G \times G^{\mathrm{op}}$-spaces. Applying geometric realization gives the $G \times G^{\mathrm{op}}$-equivariant homeomorphisms ϕ^R and γ^R.

Similarly, we obtain simplicial isomorphisms $\phi^L_\bullet : B_\bullet(G \times G^{\mathrm{op}}, G, *) \leftrightarrows B_\bullet(G, G, G) : \gamma^L_\bullet$ by

$$\phi^L_n((g, g'), g_1, \ldots, g_n) = (g, g_1, \ldots, g_n, (g_1 \cdots g_n)^{-1} g'),$$
$$\gamma^L_n(g, g_1, \ldots, g_n, g') = ((g, g_1 \cdots g_n g'), g_1, \ldots, g_n)$$

As above, these are isomorphisms simplical left $G \times G^{\mathrm{op}}$-spaces, and hence their geometric realizations give the $G \times G^{\mathrm{op}}$-equivariant homeomorphisms ϕ^R and γ^R. ∎

Combining these results, we obtain the following sequence of weak equivalences:

Proposition 4.2.8 There are weak equivalences

$$\begin{array}{ccc} EZ^* B(k, C_* G, \mathrm{Ad}(G \times G^{\mathrm{op}})) & & B(C_* G, C_* G, C_* G) \\ \simeq \Big\downarrow EZ & & \simeq \Big\downarrow EZ \\ EZ^* C_*(B(*, G, G \times G^{\mathrm{op}})) & \underset{C_*(\gamma)}{\overset{C_*(\phi)}{\rightleftarrows}} \, \cong & EZ^* C_*(B(G, G, G)) \end{array}$$

of right $C_* G^e$-modules as indicated. ∎

4.3 Comparison of Adjoint Module Structures

We now relate the $\mathrm{Ad}(X)$ C_*-module structure to the ad_0^* pullback modules we employ in Theorem 4.2.1. To do so, we employ the machinery of A_∞-algebras, and in particular morphisms between modules over an A_∞-algebra. We review the details of this theory in Appendix A.3.

We apply this theory to the adjoint modules discussed above. As before, let X be a $(G \times G^{\mathrm{op}})$-space. Note that $C_*(G^{\mathrm{op}})$ and $(C_* G)^{\mathrm{op}}$ are isomorphic DGAs, and the homotopy equivalence $EZ_{G,G^{\mathrm{op}}} : C_* G \otimes C_* G^{\mathrm{op}} \to C_*(G \times G^{\mathrm{op}})$ is a morphism of DGAs. Since $C_* X$ is a left

$C_*(G\times G^{\mathrm{op}})$-module, EZ^*C_*X is a left C_*G^e-module, and so $\mathrm{ad}_0^*\, EZ^*C_*X = (EZ\,\mathrm{ad}_0)^*C_*X$ is another left C_*G-module structure on C_*X.

As will be shown below, these two C_*G-module structures on C_*X factor through the left $C_*(G\times G)$-action $C_*(a_X)EZ_{G\times G,X}(C_*(\mathrm{id}\times i)\otimes \mathrm{id})$. Hence, we consider such $C_*(G\times G)$-modules more generally.

Proposition 4.3.1 For G, K groups and $A = C_*(G\times K)$, $\psi = EZ_{G,K}AW_{G,K} : A \to A$ is a DGA morphism.

Proof: Recall that the multiplication in A is given by $\mu = C_*((m_G\times m_K)t_{(23)})EZ_{G\times K,G\times K}$. We check that $\psi\mu = \mu(\psi\otimes\psi)$:

$$\begin{aligned}
\psi\mu &= EZ_{G,K}AW_{G,K}C_*((m_G\times m_K)t_{(23)})EZ_{G\times K,G\times K}\\
&= C_*(m_G\times m_K)EZ_{G\times G,K\times K}AW_{G\times G,K\times K}C_*(t_{(23)})EZ_{G\times K,G\times K}\\
&= C_*(m_G\times m_K)EZ_{G\times G,K\times K}(EZ_{G,G}\otimes EZ_{K,K})\tau_{(23)}(AW_{G,K}\otimes AW_{G,K})\\
&= C_*(m_G\times m_K)C_*(t_{(23)})EZ_{G,K,G,K}(AW_{G,K}\otimes AW_{G,K})\\
&= C_*((m_G\times m_K)t_{(23)})EZ_{G\times K,G\times K}(EZ_{G,K}AW_{G,K}\otimes EZ_{G,K}AW_{G,K}) = \mu(\psi\otimes\psi).
\end{aligned}$$

Furthermore, since $EZ\circ AW = \mathrm{id}$ on 0-chains, $\psi\eta = \eta$, so ψ is a DGA morphism. ∎

In light of the interpretation given above of morphisms of A_∞-modules between ordinary A-modules, the following proposition states that pullback along $EZ_{G,K}AW_{G,K}$ respects the action of $C_*(G\times K)$ only up to a system of higher homotopies.

Proposition 4.3.2 Let $A = C_*(G\times K)$, and suppose that M is a left A-module, with action a_M. Then $(EZ_{G,K}AW_{G,K})^*M$ is also a left A-module, which we denote (L, a_L). There is a quasi-isomorphism $f : L \to M$ of A_∞-modules over A, with $f_1 : L\to M$ equal to id_M.

Proof: We construct the levels f_n of this A_∞-module morphism inductively using the theory of acyclic models [41, Ch. 13]. Let $H^0 = \mathrm{id}_k$ and let $H^1 = H$, the natural homotopy with $dH + Hd = EZ\circ AW - \mathrm{id}$. By induction, we construct certain natural maps H^n of degree n of the form

$$H^n_{X_1,\dots,X_{2n}} : C_*(X_1\times X_2)\otimes\cdots\otimes C_*(X_{2n-1}\times X_{2n}) \to C_*(X_1\times\cdots\times X_{2n})$$

for spaces $X_1, \dots, X_{2n}$. Suppose that H^n has been constructed. Write $EZ_{12,34}$ for $EZ_{X_1\times X_2, X_3\times X_4}$ and so forth. Define

$$\begin{aligned}
\hat{H}^{n+1,0} &= EZ_{12,3\cdots(2n+2)}(\mathrm{id}\otimes H^n),\\
\hat{H}^{n+1,i} &= C_*(t_{(2i\,2i+1)})H^n(\mathrm{id}^{\otimes i-1}\otimes(C_*(t_{(23)})EZ_{(2i-1)(2i),(2i+1)(2i+2)})\otimes \mathrm{id}^{\otimes n-i}), \quad 0<i<n,\\
\hat{H}^{n+1,n+1} &= EZ_{1\cdots(2n),(2n+1)(2n+2)}(H^n\otimes EZ_{2n+1,2n+2}AW_{2n+1,2n+2}).
\end{aligned}$$

Let $\hat{H}^{n+1} = \sum_{i=0}^{n+1}(-1)^{i+1}\hat{H}^{n+1,i}$. A computation shows that $d\hat{H}^{n+1} = \hat{H}^{n+1}d$, so $\hat{H}^{n+1}$ is a natural chain map of degree n. By the naturality of $\hat{H}^{n+1}$, acyclic models methods apply to show that $\hat{H}^{n+1} = dH^{n+1} - (-1)^{n+1}H^{n+1}d$ for some natural map H^{n+1} of degree $n+1$, as specified above. Since

$$dH^1_{1,2} + H^1_{1,2}d = EZ_{1,2}AW_{1,2} - \mathrm{id} = \hat{H}^{1,1}_{1,2} - \hat{H}^{1,0}_{1,2} = \hat{H}^1,$$

the base case $n=0$ also satisfies the property that $dH^{n+1} - (-1)^{n+1}H^{n+1}d = \hat{H}^{n+1}$. Consequently, such natural H^n maps exist for all $n \geq 0$.

For $n \geq 1$, let $f_{n+1} = a(C_*((m_G^{n-1}\times m_K^{n-1})t_{\sigma_n})H^n_{G,K,\dots,G,K}\otimes\mathrm{id})$, where $\sigma_n \in S_{2n}$ takes $(1,\dots,2n)$ to $(1,3,\dots,2n-1,2,\dots,2n)$. By the construction of the H^n, these f_n are seen to satisfy the conditions needed for the A_∞-module morphism.

Since $f_1 = \mathrm{id}$, which is a quasi-isomorphism of chain complexes, f is a quasi-isomorphism of A_∞-modules. ∎

Corollary 4.3.3 There is a quasi-isomorphism $q : (EZ\,\mathrm{ad}_0)^*(C_*X) \to \mathrm{Ad}(X)$ of A_∞-modules over C_*G.

Proof: Note that $a = C_*(a_X)EZ_{G\times G^{\mathrm{op}},X}(C_*(\mathrm{id}\times i)\otimes\mathrm{id})$ gives $C_*(X)$ a left $C_*(G\times G)$-module structure such that the module structure of $\mathrm{Ad}(X)$ is given by $a(C_*\delta\otimes\mathrm{id})$. Furthermore, the C_*G-action a' of $(EZ\,\mathrm{ad}_0)^*(C_*X)$ is given by

$$\begin{aligned}
a' &= C_*(a_X)EZ_{G\times G^{\mathrm{op}},X}(EZ_{G,G^{\mathrm{op}}}\otimes\mathrm{id})(\mathrm{id}\otimes C_*i\otimes\mathrm{id})(AW_{G,G}\otimes\mathrm{id})(C_*\delta\otimes\mathrm{id})\\
&= C_*(a_X)EZ_{G\times G^{\mathrm{op}},X}(C_*(\mathrm{id}\times i)\otimes\mathrm{id})(EZ_{G,G}\otimes\mathrm{id})(AW_{G,G}\otimes\mathrm{id})(C_*\delta\otimes\mathrm{id})\\
&= a((EZ_{G,G}AW_{G,G}C_*\delta)\otimes\mathrm{id}).
\end{aligned}$$

The proposition above then applies to the $C_*(G \times G)$-module structures a and $(EZ \circ AW)^* a$ on C_*X to yield an A_∞ quasi-isomorphism. Pulling this morphism back along the DGA morphism $C_*\delta : C_*G \to C_*(G \times G)$ yields the desired quasi-isomorphism of A_∞-modules over C_*G. ∎

Consequently, this quasi-isomorphism of A_∞-modules over C_*G induces a quasi-isomorphism of chain complexes $B(k, C_*G, (EZ\,\mathrm{ad}_0)^*(C_*X)) \to B(k, C_*G, \mathrm{Ad}(X))$. A similar argument shows that there exists a quasi-isomorphism $(EZ\,\mathrm{ad}_0)^* C_*X \to \mathrm{Ad}(X)$ of right A_∞-modules for X with a right $G \times G^{\mathrm{op}}$-action. Connecting these isomorphisms yields the following:

Theorem 4.3.4 $B(k, C_*G, \mathrm{ad}_0^* C_*G^e)$ and $B(C_*G, C_*G, C_*G)$ are homotopy equivalent as right C_*G^e-modules.

Proof: Proposition 4.2.8, Corollary 4.3.3, and the quasi-isomorphism $EZ : C_*G^e \to C_*(G \times G^{\mathrm{op}})$ combine to produce the following diagram of weak equivalences and isomorphisms of right $C_*(G)^e$-modules:

$$\begin{array}{ccc}
B(k, C_*G, \mathrm{ad}_0^* C_*G^e) & & B(C_*G, C_*G, C_*G) \\
\simeq \downarrow B(\mathrm{id},\mathrm{id},EZ) & & \\
EZ^* B(k, C_*G, (EZ\,\mathrm{ad}_0)^* C_*(G \times G^{\mathrm{op}})) & & \simeq \downarrow EZ \\
\simeq \downarrow B(\mathrm{id},\mathrm{id},q) & & \\
EZ^* B(k, C_*G, \mathrm{Ad}(G \times G^{\mathrm{op}})) & & \\
\simeq \downarrow EZ & & \\
EZ^* C_*(B(*, G, G \times G^{\mathrm{op}})) & \underset{C_*(\gamma)}{\overset{C_*(\phi)}{\rightleftarrows}}\ \cong & EZ^* C_*(B(G, G, G))
\end{array}$$

Consequently, $B(k, C_*G, \mathrm{ad}_0^* C_*G^e)$ and $B(C_*G, C_*G, C_*G)$ are related by a zigzag of weak equivalences of C_*G^e-modules. Since they are both semifree, and hence cofibrant, C_*G^e-modules, the remarks in Section 2.2.2 imply that they are in fact homotopy equivalent. ∎

We now complete the proof of Theorem 4.1.1.

Proof (Theorem 4.1.1): By Corollary 4.2.2, Corollary 4.3.3, and the naturality of this extended functoriality of Ext and Tor with respect to evaluation, we obtain the diagram

$$\begin{array}{ccccc} CH^*(C_*\Omega X) & \xrightarrow{\simeq} & H^\bullet(X; \mathrm{ad}_0\, C_*\Omega X^e)) & \xrightarrow[\simeq]{q_*} & H^\bullet(X; \mathrm{Ad}(\Omega X)) \\ & & \simeq \big\downarrow \mathrm{ev}_{[X]} & & \simeq \big\downarrow \mathrm{ev}_{[X]} \\ CH_{*+d}(C_*\Omega X) & \xrightarrow{\simeq} & H_{\bullet+d}(X; \mathrm{ad}_0\, C_*\Omega X^e)) & \xrightarrow[\simeq]{q_*} & H_{\bullet+d}(X; \mathrm{Ad}(\Omega X)) \end{array}$$

of weak equivalences. The outside of the diagram then provides the diagram of Theorem 4.1.1.∎

We note also that these techniques extend the multiplication map $\mu : \mathrm{Ad}(G) \otimes \mathrm{Ad}(G) \to \mathrm{Ad}(G)$ to an A_∞ map on $\mathrm{Ad}(G)$ that is compatible with the comultiplication on C_*G:

Proposition 4.3.5 There is a morphism of A_∞-modules $\tilde{\mu} : \Delta^*(\mathrm{Ad}(G) \otimes \mathrm{Ad}(G)) \to \mathrm{Ad}(G)$ with $\tilde{\mu}_1 = \mu : C_*G \otimes C_*G \to C_*G$.

Proof: Define $\tilde{\mu}$ as the following composite of A-module and A_∞-A-module morphisms, where f is the A_∞-module morphism from Prop. 4.3.2:

$$\begin{aligned} \Delta^*(\mathrm{Ad}(G) \otimes \mathrm{Ad}(G)) &\xrightarrow{EZ} \Delta^* EZ^*(C_*(G^c \times G^c)) = C_*(\delta)^*(EZ \circ AW)^*(C_*(G^c \times G^c)) \\ &\xrightarrow{f} C_*(\delta)^*(C_*(G^c \times G^c)) = C_*(\delta^*(G^c \times G^c)) \\ &\xrightarrow{C_*(m)} C_*(G^c) \end{aligned}$$

Since $f_1 = \mathrm{id}$, $\tilde{\mu}_1 = C_*(m) \circ EZ = \mu$. ∎

We use this multiplication map in Chapter 5 to related the D isomorphism to a suitable notion of cap product in Hochschild homology and cohomology.

Chapter 5

BV Algebra Structures

5.1 Multiplicative Structures

As before, now let M be a closed, k-oriented manifold of dimension d. We now investigate whether the isomorphism between $H_{*+d}(LM)$ and $HH^*(C_*\Omega M)$ established in Corollary 4.1.2 is one of rings, taking the Chas-Sullivan product on $H_{*+d}(LM)$ to the Hochschild cup product on $HH^*(C_*\Omega M)$. To do so, we must examine a homotopy-theoretic construction of the Chas-Sullivan product on the spectrum LM^{-TM} and relate it to the ring spectrum structure of the topological Hochschild cohomology of the suspension spectrum $S[\Omega M]$. Again treating ΩM as a topological group G, we use the function spectrum $F_G(EG_+, S[G^c])$ as an intermediary, and we adapt some of the techniques of Abbaspour, Cohen, and Gruher [1] and Cohen and Klein [8] to compare the ring spectrum structures. Smashing with the Eilenberg-Mac Lane spectrum and passing back to the derived category of chain complexes over k then recovers our earlier chain-level equivalences.

5.2 Fiberwise Spectra and Atiyah Duality

We review some of the fundamental constructions and theorems in the theory of fiberwise spectra discussed in [8].

Definition 5.2.1 Let X be a topological space, and let Top/X be the category of spaces over

X. Let R_X be the category of retractive spaces over X: such a space Y has maps $s_Y : X \to Y$ and $r_Y : Y \to X$ such that $r_Y s_Y = \mathrm{id}_X$. Both categories are enriched over Top. Furthermore, there is a forgetful functor $u_X : \mathsf{R}_X \to \mathsf{Top}/X$ with a left adjoint v_X given by $v_X(Y) = Y \amalg X$, with $X \to Y \amalg X$ the inclusion. We often denote $v_X(Y)$ as Y_+ when X is clear from context.

Given $Y \in \mathsf{Top}/X$, its *unreduced fiberwise suspension* $S_X Y$ is the double mapping cylinder $X \cup Y \times I \cup X$; note this determines a functor $S_X : \mathsf{Top}/X \to \mathsf{Top}/X$. Given $Y \in \mathsf{R}_X$, its *(reduced) fiberwise suspension* $\Sigma_X Y$ is $S_X Y \cup_{S_X X} X$. This construction also determines a functor $\Sigma_X : \mathsf{R}_X \to \mathsf{R}_X$.

Given $Y, Z \in \mathsf{R}_X$, define their *fiberwise smash product* $Y \wedge_X Z$ as the pushout of $X \leftarrow Y \cup_X Z \to Y \times_X Z$, where the map $Y \cup_X Z \to Y \times_X Z$ takes y to $(y, s_Z r_Y y)$ and z to $(s_Y r_Z z, z)$. The fiberwise smash product then defines a functor $\wedge_X : \mathsf{R}_X \times \mathsf{R}_X \to \mathsf{R}_X$ making R_X a symmetric monoidal category, with unit $S^0 \times X$. Furthermore, $\Sigma_X Y \cong (S^1 \times X) \wedge_X Y$. ▪

The notion of fiberwise reduced suspension is key in constructing spectra fibered over X.

Definition 5.2.2 A *fibered spectrum* E over X is a sequence of objects $E_j \in \mathsf{R}_X$ for $j \in \mathbb{N}$ together with maps $\Sigma_X E_j \to E_{j+1}$ in R_X.

Given $Y \in \mathsf{R}_X$, its fiberwise suspension spectrum is the spectrum $\Sigma_X^\infty Y$ with jth space defined by $\Sigma_X^j Y$, with the structure maps given by the identification $\Sigma_X(\Sigma_X^j Y) \cong \Sigma_X^{j+1} Y$. ▪

Spectra fibered over X form a model category, with notions of weak equivalences, cofibrations, and fibrations arising as in the context of traditional spectra (i.e., spectra fibered over a point $*$). Given a spectrum E fibered over X, one can produce covariant and contravariant functors from Top/X to spectra as follows.

Definition 5.2.3 Let E be a spectrum fibered over X, and take $Y \in \mathsf{Top}/X$. Assume E to be fibrant in the model structure of such fibered spectra. Define the spectrum $H_\bullet(Y; E)$ to be the homotopy cofiber of the map $Y \to Y \times_X E$. Define $H^\bullet(Y; E)$ levelwise to be the space $\mathrm{Hom}_{\mathsf{Top}/X}(Y^c, E_j)$ in level j, where Y^c is a functorial cofibrant replacement for Y in the category Top/X.

Since both of these constructions are functorial in Y, they determine functors $H_\bullet(-; E)$ and $H^\bullet(-; E)$ which we call *homology* and *cohomology with E-coefficients.* ▪

We use these notions of spectrum-valued homology and cohomology functors to express a form of Poincaré duality. First, however, we must explain how to twist a fibered spectrum over X by a vector bundle over X.

Definition 5.2.4 Let E be a spectrum fibered over X and let ξ be a vector bundle over X. Define the *twist of E by* ξ, ${}^{\xi}E$, levelwise by $({}^{\xi}E)_j = S^{\xi} \wedge_X E_j$, where S^{ξ} is the sphere bundle over X given by one-point compactification of the fibers of ξ. By introducing suspensions appropriately, the twist of E by a virtual bundle ξ is defined analogously. ▪

Poincaré or Atiyah duality can now be expressed in the following form:

Theorem 5.2.5 Let N be a closed manifold of dimension d with tangent bundle TN, and let $-TN$ denote the virtual bundle of dimension $-d$ representing the stable normal bundle of N. Let E be a spectrum fibered over N. Then there is a weak equivalence of spectra

$$H_{\bullet}(N; {}^{-TN}E) \simeq H^{\bullet}(N; E). \tag{5.2.6}$$

Furthermore, this equivalence is natural in E. ▪

5.2.1 The Chas-Sullivan Loop Product

We recall a homotopy-theoretic construction of the Chas-Sullivan loop product from [7] in terms of umkehr maps on generalized Thom spectra, and we then illustrate how this loop product is expressed in [8] using fiberwise spectra and fiberwise Atiyah duality.

Let M be a smooth, closed d-manifold, and note that LM is a space over M via the evaluation map at $1 \in S^1$, $\mathrm{ev} : LM \to M$. Let $L_\infty M$ be the space of maps of the figure-eight, $S^1 \vee S^1$, into M. Then

$$\begin{array}{ccc} L_\infty M & \xrightarrow{\tilde{\Delta}} & LM \times LM \\ \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev} \times \mathrm{ev}} \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

is a pullback square. Furthermore, the basepoint-preserving pinch map $S^1 \to S^1 \vee S^1$ induces a map γ of spaces over M:

$$\begin{array}{ccc} L_\infty M & \xrightarrow{\gamma} & LM \\ \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}} \\ M & = & M \end{array}$$

Since the map $\tilde{\Delta}$ is the pullback of a finite-dimensional embedding of manifolds, it induces a collapse map

$$\Delta^! : (LM \times LM)_+ \to L_\infty M^{\nu_\Delta},$$

where ν_Δ here is the pullback along ev of the normal bundle ν_Δ to the embedding $\Delta : M \to M \times M$. This normal bundle is isomorphic to TM, the tangent bundle to M. This collapse map is compatible with the formation of the Thom spectra of a stable vector bundle ξ on $LM \times LM$. Taking $\xi = -TM \times -TM$, and noting that $\Delta^*(-TM \times -TM) = -TM \oplus -TM$, this gives an umkehr map

$$\Delta^! : (LM \times LM)^{-TM \times -TM} \to L_\infty M^{-TM}.$$

Composing $\Delta^!$ with the smash product map $LM^{-TM} \wedge LM^{-TM} \to (LM \times LM)^{-TM \times -TM}$ and the map $\gamma^{-TM} : L_\infty M^{-TM} \to LM^{-TM}$ induced by γ gives a homotopy-theoretic construction of the loop product

$$\circ : LM^{-TM} \wedge LM^{-TM} \to LM^{-TM}.$$

A k-orientation of M induces a Thom isomorphism $LM^{-TM} \wedge Hk \cong \Sigma^{-d}\Sigma^\infty LM_+ \wedge Hk$, so passing to spectrum homotopy groups gives the loop product on homology with the expected degree shift.

We now consider this loop product from the perspective of fiberwise spectra. Since $\mathrm{ev} : LM \to M$ makes LM a space over M, $LM_+ = LM \amalg M$ is a retractive space over M, and iterated fiberwise suspensions of LM over M produce a fiberwise spectrum $\Sigma^\infty_M LM_+$ over M. Recall from Definition 5.2.3 that for a spectrum E fibered over X and a space Y over X, $H_\bullet(Y; E) = (Y \times_X E) \cup CY$ and $H^\bullet(Y; E)$ is the spectrum of maps $\mathrm{Map}_X(Y, E^f)$ of Y over X into a fibrant replacement for E.

Proposition 5.2.7 ([8]) As spectra, $LM^{-TM} \simeq H_\bullet(M; {}^{-TM}\Sigma^\infty_M LM_+)$. By fiberwise Atiyah duality,

$$LM^{-TM} \simeq H^{\bullet}(M; \Sigma_M^{\infty} LM_+).$$

Since LM is a fiberwise A_∞-monoid over M, $\Sigma_M^\infty LM_+$ is a fiberwise A_∞-ring spectrum, and so the spectrum of sections $H^\bullet(M; \Sigma_M^\infty LM_+)$ is also a ring spectrum. The Chas-Sullivan loop product on LM^{-TM} arises as the induced product on LM^{-TM}.

Proof: We check that $LM^{-TM} \simeq H_\bullet(M; {}^{-TM}\Sigma_M^\infty LM_+)$. Let ν be an $(L-d)$-dimensional normal bundle for M. The $(j+L)$th space of LM^{-TM} is then the Thom space $LM^{\mathrm{ev}^*(\nu)\oplus\epsilon^j}$, the one-point compactification of $\mathrm{ev}^*(\nu)\oplus\epsilon^j$. The $(j+L)$th space of ${}^{-TM}\Sigma_M^\infty LM_+$ is $S^j \wedge S^\nu \wedge_M LM_+$, which is seen to be the fiberwise compactification of $LM^{\mathrm{ev}^*(\nu)\oplus\epsilon^j}$ over M. Applying $H_\bullet(M; -)$ attaches the cone CM to this space along the M-section of basepoints added by the fiberwise compactification, thus making a space homotopy equivalent to $LM^{\mathrm{ev}^*(\nu)\oplus\epsilon^j}$.

Fiberwise Atiyah duality then shows that $LM^{-TM} \simeq H^\bullet(M; \Sigma_M^\infty LM_+)$.

We compare each step of the original LM^{-TM} construction of the loop product to the ring spectrum structure on $H^\bullet(M; \Sigma_M^\infty LM_+)$. First, since $\Sigma_M^\infty LM_+ \wedge \Sigma_M^\infty LM_+ \cong \Sigma_{M\times M}^\infty L(M\times M)_+$ as spectra fibered over $M \times M$, the square

$$\begin{array}{ccc} LM^{-TM} \wedge LM^{-TM} & \xrightarrow{\wedge} & (L(M\times M))^{-T(M\times M)} \\ \downarrow \simeq & & \downarrow \simeq \\ \Gamma(\Sigma_M^\infty LM_+) \wedge \Gamma(\Sigma_M^\infty LM_+) & \xrightarrow{\wedge} & \Gamma(\Sigma_{M\times M}^\infty L(M\times M)_+) \end{array}$$

commutes. Next, pullback along Δ induces a map of spectra

$$\Delta^\bullet : H^\bullet(M\times M, \Sigma_{M\times M}^\infty L(M\times M)_+) \to H^\bullet(M, \Sigma_{M\times M}^\infty L(M\times M)_+).$$

The universal property of the pullback $L_\infty M$ induces a homeomorphism $\mathrm{Map}_{M\times M}(M, L(M\times M)_+) \cong \mathrm{Map}_M(M, L_\infty M_+)$, and thus an equivalence

$$H^\bullet(M, \Sigma_{M\times M}^\infty L(M\times M)_+) \simeq H^\bullet(M, \Sigma_M^\infty L_\infty M_+).$$

Hence, the umkehr map diagram

$$\begin{array}{ccc} (LM \times LM)^{-T(M\times M)} & \xrightarrow{\Delta^!} & L_\infty M^{-TM} \\ \downarrow \simeq & & \downarrow \simeq \\ H^\bullet(M, \Sigma^\infty_{M\times M} L(M\times M)_+) & \xrightarrow{\Delta^\bullet} & H^\bullet(M, \Sigma^\infty_M L_\infty M_+) \end{array}$$

commutes. Finally, by the naturality of fiberwise Atiyah duality in the spectrum argument, the diagram

$$\begin{array}{ccc} L_\infty M^{-TM} & \xrightarrow{\gamma^{-TM}} & LM^{-TM} \\ \downarrow \simeq & & \downarrow \simeq \\ H^\bullet(M, \Sigma^\infty_M L_\infty M_+) & \xrightarrow{\gamma^\bullet} & H^\bullet(M, \Sigma^\infty_M LM_+) \end{array}$$

commutes. ∎

5.2.2 Ring Spectrum Equivalences

For notational simplicity, let G be a topological group replacement for ΩX. Furthermore, if Y is an unbased space, we follow Klein [26] in letting $S[Y]$ denote the fibrant replacement of the suspension spectrum of Y_+. Thus, the jth space of $S[Y]$ is $Q(S^j \wedge Y_+)$, where $Q = \Omega^\infty\Sigma^\infty$ is the stable homotopy functor. Furthermore, we let E^f denote a fibrant replacement for a spectrum E fibered over a space Z; if the fibers are suspension spectra $\Sigma^\infty Y_+$, then the fibers of E^f may be taken to be $S[Y]$.

We establish spectrum-level analogues of the Goodwillie isomorphism BFG and the isomorphism $\Lambda_*(G, M)$.

Proposition 5.2.8 There are equivalences of spectra $\Gamma : S[LM] \to S[G] \wedge_G EG_+$ and $\Lambda_\bullet : S[G] \wedge_G EG_+ \to THH^S(S[G])$.

Proof: We first establish the equivalence Γ. Since $G \simeq \Omega M$, $M \simeq BG$. Furthermore, $LM \simeq LBG$ over this equivalence, and so $\Sigma^\infty LM_+ \simeq \Sigma^\infty LBG_+$. Next, the well-known homotopy equivalence $LBG \simeq G^c \times_G EG$ shows that

$$\Sigma^\infty LBG_+ \simeq \Sigma^\infty (G^c \times_G EG)_+.$$

Passing to fibrant replacements then gives $S[LM] \simeq S[G^c] \wedge_G EG_+$.

Take $B(G, G, *) = |B_\bullet(G, G, *)|$ as a model for EG. Then

$$W_n = S[G^c] \wedge_G B_n(G, G, *)_+$$

determines a simplicial spectrum with $|W_\bullet| = S[G^c] \wedge_G EG_+$. Likewise,

$$V_n = S[G] \wedge_{G\times G^{\mathrm{op}}} B_n(G, G, G)_+$$

determines a simplicial spectrum such that $|V_\bullet| \simeq THH^S(S[G])$. Hence, we show there is an isomorphism $\chi_\bullet : W_\bullet \xrightarrow{\cong} V_\bullet$ of simplicial spectra. In fact, this map is the composite of the isomorphism

$$S[G^c] \wedge_G B_n(G, G, *)_+ \cong S[G] \wedge_{G\times G^{\mathrm{op}}} B_n(G \times G^{\mathrm{op}}, G, *)_+$$

and $S[G] \wedge_{G\times G^{\mathrm{op}}} \phi^L_\bullet$, where $\phi^L_\bullet$ is the simplicial homeomorphism $B_\bullet(G \times G^{\mathrm{op}}, G, *) \to B_\bullet(G, G, G)$ of Proposition 4.2.7. Explicitly, the χ_n are given by

$$\chi_n(a \wedge [g_1 \mid \cdots \mid g_n]) = (g_1 \cdots g_n)a \wedge [g_1 \mid \cdots \mid g_n].$$

∎

We also produce spectrum-level analogues of the weak equivalences among $C_{*+d}(LM)$, $R\operatorname{Hom}^*_{C_*G}(k, \mathrm{Ad}(G))$, and $CH^*(C_*G)$. Westerland has shown [48] that $F_G(EG_+, \Sigma^\infty G^c_+)$ is a ring spectrum for G a general topological group, and the topological Hochschild cohomology $THH_S(S[G])$ of the ring spectrum $S[G]$ is likewise well-known to be a ring spectrum itself. The composite isomorphism should be equivalent to Klein's [27] equivalence of spectra $(LX)^{-\tau_X} \simeq THH_S(S[\Omega X])$ for a Poincaré duality space (X, τ_X), although we make more of the ring structure explicit here.

We first relate LM^{-TM} and $F_G(EG_+, S[G^c])$ as ring spectra.

Proposition 5.2.9 There is an equivalence of ring spectra $\Psi : LM^{-TM} \to F_G(EG_+, S[G^c])$.

Proof: Recall from Proposition 5.2.7 that the spectrum

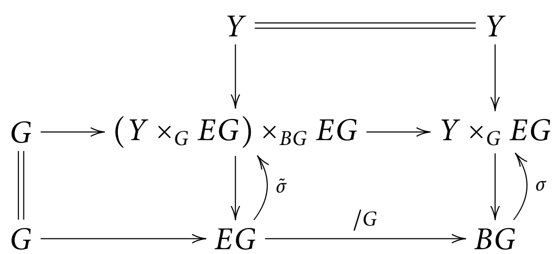
$$LM^{-TM} \simeq H_\bullet(M; {}^{-TM}\Sigma^\infty_M LM) \simeq H^\bullet(M; \Sigma^\infty_M LM).$$

Since $LM \simeq LBG$ over the equivalence $M \simeq BG$, $H^\bullet(M; \Sigma^\infty_M LM)$ and $H^\bullet(BG; \Sigma^\infty_{BG} LBG)$ are equivalent spectra. Since $LBG \simeq G^c \times_G EG$, this is equivalent to $H^\bullet(BG; \Sigma^\infty_{BG}(G^c \times_G EG)_+)$. The jth level of the target spectrum $\Sigma^\infty_{BG}(G^c \times_G EG)_+$ is the space

$$(S^j \times BG) \wedge_{BG} ((G^c \times_G EG) \amalg BG) \cong (S^j \wedge G^c_+) \times_G EG.$$

Since the $H^\bullet$ construction implicitly performs a fibrant replacement on its target, the jth space of the spectrum $H^\bullet(BG; \Sigma^\infty_{BG}(G^c \times_G EG)_+)$ is $\mathrm{Map}_{BG}(BG, Q(S^j \wedge G^c_+) \times_G EG)$.

For a G-space Y, to pass from $\mathrm{Map}_{BG}(BG, Y \times_G EG)$ to $\mathrm{Map}_G(EG, Y)$, we form the pullback diagram



Note that $(Y \times_G EG) \times_{BG} EG$ has a left G-action coming from the right EG factor. By pullback, $\sigma \in \mathrm{Map}_{BG}(BG, Y \times_G EG)$ determines a section $\tilde\sigma \in \mathrm{Map}_{EG}(EG, (Y \times_G EG) \times_{BG} EG)$, with $\tilde\sigma(e) = (\sigma([e]), e)$. Since $\tilde\sigma(ge) = (\sigma([ge]), ge) = g \cdot (\sigma([e]), e)$, $\tilde\sigma$ is G-equivariant.

Since EG is a free G-space, $(Y \times_G EG) \times_{BG} EG$ is homeomorphic to $Y \times EG$ by $([y, e], e) \mapsto (y, e)$. Furthermore, $Y \times EG$ has a left G-action given by $\Delta^*_G(i^* Y \times EG)$, where the pullback i^* by the inverse map i for G converts the right G-space Y into a left G-space. Since

$$g \cdot ([y, e], e) = ([yg^{-1}, ge], ge) \mapsto (yg^{-1}, ge) = g \cdot (y, e),$$

this homeomorphism is G-equivariant with the above left G-action on $(Y \times_G EG) \times_{BG} EG$. Hence, $\tilde\sigma$ corresponds to a G-equivariant section $\sigma' \in \mathrm{Map}_{EG}(EG, Y \times EG)$. Since $Y \times EG$ is a product, σ' is determines by the projections $\pi_{EG} \circ \sigma' = \mathrm{id}_{EG}$ and $\pi_Y \circ \sigma'$, both of which are G-equivariant maps. Consequently, we obtain the homeomorphism $\mathrm{Map}_{BG}(BG, Y \times_G EG) \cong \mathrm{Map}_G(EG, Y)$.

Applying this correspondence levelwise with $Y = Q(S^j \wedge G^c_+)$, this space of sections is homeomorphic to $\mathrm{Map}_G(EG, Q(S^j \wedge G^c_+))$, the jth space of $F_G(EG_+, S[G^c])$.

We now show that under these equivalences, the product on LM^{-TM} coincides with that on $F_G(EG_+, S[G^c])$. From above, the product on LM^{-TM} is equivalent to that on $H^\bullet(M; \Sigma^\infty_M LM_+)$, given by

$$H^\bullet(M; \Sigma^\infty_M LM_+)^{\wedge 2} \xrightarrow{\wedge} H^\bullet(M \times M; \Sigma^\infty_{M\times M}(LM \times LM)_+) \xrightarrow{\Delta^\bullet_M} H^\bullet(M; \Sigma^\infty_M (L_\infty M)_+) \xrightarrow{\gamma_*} H^\bullet(M; \Sigma^\infty_M (LM)_+).$$

Since $LM \simeq LBG$ over $M \simeq BG$, and since $LBG \simeq G^c \times_G EG$ as fiberwise monoids over BG [19, App. A], this sequence is equivalent to

$$H^\bullet(BG; \Sigma^\infty_{BG}(G^c \times_G EG)_+)^{\wedge 2} \xrightarrow{\wedge} H^\bullet(B(G\times G), \Sigma^\infty_{B(G\times G)}((G^c \times G^c) \times_{G\times G} E(G\times G))_+) \xrightarrow{B(\Delta_G)^\bullet} H^\bullet(BG, \Sigma^\infty_{BG}(\Delta^*_G(G^c \times G^c) \times_G EG)_+) \xrightarrow{\mu_*} H^\bullet(BG, \Sigma^\infty_{BG}(G^c \times_G EG)_+)$$

Finally, passing to equivariant maps into the fibers, this sequence is equivalent to

$$F_G(EG_+, S[G^c])^{\wedge 2} \xrightarrow{\wedge} F_{G\times G}(E(G\times G)_+, S[G^c \times G^c]) \xrightarrow{E(\Delta_G)^* \circ \Delta^*_G} F_G(EG_+, S[\Delta^*_G(G^c \times G^c)])) \xrightarrow{S[\mu]_*} F_G(EG_+, S[G^c]).$$

This is the descripton given by Westerland [48] of the ring structure of $F_G(EG_+, S[G^c])$. ▪

We now relate $F_G(EG_+, S[G^c])$ and $THH_S(S[G])$ as ring spectra.

Proposition 5.2.10 There is an equivalence of ring spectra $\Lambda^\bullet(G) : F_G(EG_+, S[G^c]) \to THH_S(S[G])$.

Proof: We first show that $F_G(EG_+, S[G^c]) \simeq THH_S(S[G])$ as spectra. As above, take $B(G, G, *)$ as a model of EG with the usual left G-action. Let

$$Z^n = \mathrm{Map}_G(B_n(G, G, *), S[G^c])$$

be the corresponding cosimplicial spectrum; then $F_G(EG_+, S[G^c]) = \mathrm{Tot}\, Z^\bullet$. Similarly, the endomorphism operad $F_S(S[G]^{\wedge\bullet}, S[G])$ of $S[G]$ is an operad with multiplication on account of the unit and multiplication maps of $S[G]$, and so by results of McClure and Smith [32], it admits a canonical cosimplicial structure. Furthermore, its totalization is $THH_S(S[G])$. Let

$$Y^n = \mathrm{Map}(G^n, S[G]),$$

which is equivalent to $F_S(S[G]^{\wedge n}, S[G])$. Then $\mathrm{Tot}\, Y^\bullet \simeq THH_S(S[G])$. Consequently, we need only exhibit an isomorphism $\psi^\bullet : Z^\bullet \to Y^\bullet$ of cosimplicial spectra.

Since $G^c = \mathrm{ad}^* G$, the equivariance adjunction between G-spaces and $G \times G^{\mathrm{op}}$-spaces followed by pullback along the simplicial homeomorphism $\gamma^L_\bullet$ of Proposition 4.2.7 gives that

$$\mathrm{Map}_G(B_\bullet(G, G, *), S[G^c]) \cong \mathrm{Map}_{G\times G^{\mathrm{op}}}(B_\bullet(G, G, G), S[G])$$

as cosimplicial spectra. Finally, by the freeness of each $B_q(G, G, G)$ as a $(G \times G^{\mathrm{op}})$-space,

$$\mathrm{Map}_{G\times G^{\mathrm{op}}}(B_\bullet(G, G, G), S[G]) \cong \mathrm{Map}(G^\bullet, S[G]),$$

also as cosimplicial spectra. Explicitly, for $a \in Z^p$, we have that

$$\psi^p(a)([g_1 \mid \cdots \mid g_p]) = a([g_1 \mid \cdots \mid g_p])(g_1 \cdots g_p).$$

We now show this equivalence is one of ring spectra. Since the cosimplicial structure of $Y^\bullet$ comes from an operad with multiplication, it has a canonical cup-pairing $Y^p \wedge Y^q \to Y^{p+q}$ coming from the operad composition maps and the multiplication. Therefore, $\mathrm{Tot}\, Y^\bullet \simeq THH_S(S[G])$ is an algebra for an operad $\mathcal{C}$ weakly equivalent to the little 2-cubes operad. Hence, $THH_S(S[G])$ is an E_2-ring spectrum, and *a fortiori* an A_∞-ring spectrum.

We can also describe the product on $F_G(EG_+, S[G^c])$ cosimplicially. The diagonal map Δ_G gives a canonical G-equivariant diagonal map $\Delta : B_\bullet(G, G, *) \to B_\bullet(G, G, *) \times B_\bullet(G, G, *)$; on realizations, this gives the G-equivariant map $E(\Delta_G) : EG \to EG \times EG$. This diagonal map induces a sequence of cosimplicial maps

$$\begin{aligned}&\mathrm{Map}_G(B_q(G, G, *), S[G^c]) \wedge \mathrm{Map}_G(B_q(G, G, *), S[G^c]) \\ &\quad \to \mathrm{Map}_{G\times G}(B_q(G, G, *) \times B_q(G, G, *), S[G^c \times G^c]) \to \mathrm{Map}_G(B_q(G, G, *), S[G^c])\end{aligned}$$

when composed with pullback along Δ_G and with $S[\mu]$. These assemble to a cosimplicial multiplication map $\mathrm{Tot}\, Z^\bullet \wedge \mathrm{Tot}\, Z^\bullet \to \mathrm{Tot}\, Z^\bullet$, which produces the strictly associative product map on $F_G(EG_+, S[G^c])$.

This cosimplicial map induces a canonical cup-pairing $\cup : Z^p \wedge Z^q \to Z^{p+q}$, and as a result there is a map $\mathrm{Tot}\, Z^\bullet \wedge \mathrm{Tot}\, Z^\bullet \to \mathrm{Tot}\, Z^\bullet$ for each u with $0 < u < 1$. Each such map is also homotopic to the strict multiplication on $\mathrm{Tot}\, Z^\bullet$. Furthermore, these maps assemble into an action of the little 1-cubes operad on $\mathrm{Tot}\, Z^\bullet$, making it an A_∞-ring spectrum.

Consequently, we need to show that the isomorphism $\psi^\bullet : Z^\bullet \to Y^\bullet$ of cosimplicial spectra induces an isomorphism of these two cup-pairings. We do that explicitly using the definition of $\psi^\bullet$ and these cup-pairings. If $a \in Z^p$ and $b \in Z^q$, then

$$\begin{aligned}(a \cup b)(g[g_1 \mid \cdots \mid g_{p+q}]) &= a(g[g_1 \mid \cdots \mid g_p])b(gg_1 \cdots g_p[g_{p+1} \mid \cdots \mid g_{p+q}]) \\ &= ga([g_1 \mid \cdots \mid g_p])g_1 \cdots g_p b([g_{p+1} \mid \cdots \mid g_{p+q}])(g_1 \cdots g_p)^{-1} g^{-1}.\end{aligned}$$

Hence,

$$\begin{aligned}(\psi^p(a) \cup \psi^q(b))([g_1 \mid \cdots \mid g_{p+q}]) &= \psi(a)([g_1 \mid \cdots \mid g_p])\psi(b)([g_{p+1} \mid \cdots \mid g_{p+q}]) \\ &= a([g_1 \mid \cdots \mid g_p])(g_1 \cdots g_p)b([g_{p+1} \mid \cdots \mid g_{p+q}])(g_{p+1} \cdots g_{p+q}) \\ &= (a \cup b)([g_1 \mid \cdots \mid g_{p+q}])(g_1 \cdots g_{p+q}) \\ &= \psi^{p+q}(a \cup b)([g_1 \mid \cdots \mid g_{p+q}]).\end{aligned}$$

Consequently, $\psi^p(a) \cup \psi^q(b) = \psi^{p+q}(a \cup b)$, so ψ induces an isomorphism of cup-pairings, as desired. We conclude that the A_∞-ring structures on $F_G(EG_+, S[G^c])$ and $THH_S(S[G])$ are equivalent. ∎

Naturally, we want to connect these equivalences of spectra to the quasi-isomorphisms of k-chain complexes determined above. Smashing these spectra with Hk, the Eilenberg-Mac Lane spectrum of k, and using the smallness of EG as a G-space when M is a Poincaré duality space,

$$F_G(EG_+, S[G^c]) \wedge Hk \simeq F_G(EG_+, S[G^c] \wedge Hk) \simeq F_{G_+ \wedge Hk}(EG_+ \wedge Hk, S[G^c] \wedge Hk)$$

and $THH_S(S[G]) \simeq THH_{Hk}(S[G] \wedge Hk)$. Thus,

$$LM^{-TM} \wedge Hk \simeq F_{S[G]\wedge Hk}(EG_+ \wedge Hk, S[G^c] \wedge Hk) \simeq THH_{Hk}(S[G] \wedge Hk).$$

Furthermore, since M is k-oriented, $LM^{-TM} \wedge Hk \simeq \Sigma^{-d} S[LM]$ by the Thom isomorphism.

We relate these spectrum-level constructions back to the chain-complex picture above. By results of Shipley [38], there is a zigzag of Quillen equivalences between the model categories of Hk-algebras and DGAs over k; the derived functors between the homotopy categories are denoted $\Theta : Hk\text{-alg} \to \mathrm{DGA}/k$ and $\mathbb{H} : \mathrm{DGA}/k \to Hk\text{-alg}$. Furthermore, this correspondence induces Quillen equivalences between A-Mod and $\mathbb{H}A$-Mod for A a DGA over k, and between B-Mod and ΘB-Mod for B an Hk-algebra.

We also have that $\Theta(S[G] \wedge Hk)$ is weakly equivalent to $C_*(G; k)$, and so their categories of modules are also Quillen equivalent, since $\mathrm{Ch}(k)$ exhibits Quillen invariance for modules [37, 3.11]. Consequently, the categories of modules over $S[G] \wedge Hk$ and over $C_*(G; k)$ are Quillen equivalent. Since $EG_+ \wedge Hk$ is equivalent to $C_*(EG; k) \simeq k$, the equivalence above gives the quasi-isomorphisms

$$C_{*+d}(LM) \simeq R\operatorname{Hom}_{C_*\Omega M}(k, \mathrm{Ad}(\Omega M)) \simeq CH^*(C_*\Omega M)$$

we developed above. In paticular, we recover the derived Poincaré duality map as the composite of the Atiyah duality map and the Thom isomorphism. Applying H_*, we recover the isomorphisms

$$H_{*+d}(LM) \simeq \operatorname{Ext}^*_{C_*\Omega M}(k, \mathrm{Ad}(\Omega M)) \simeq HH^*(C_*\Omega M).$$

We summarize these results in the following theorem:

Theorem 5.2.11 Any model for $R\operatorname{Hom}_{C_*\Omega M}(k, \mathrm{Ad}(\Omega M))$ is an algebra up to homotopy (i.e., a monoid in $\operatorname{Ho}\mathrm{Ch}(k)$) coming from the ring spectrum structure of $F_{\Omega M}(E\Omega M_+, S[\Omega M^c])$. Furthermore, this algebra is equivalent to the A_∞-algebra $CH^*(C_*\Omega M)$, and in homology induces the loop product on $H_{*+d}(LM)$.

Therefore, the isomorphism $BFG \circ D : HH^*(C_*\Omega M) \to H_{*+d}(LM)$ is one of graded algebras, taking the Chas-Sullivan loop product to the Hochschild cup product. ∎

In particular, the model $\mathrm{Hom}_A(B(k,A,A),\mathrm{Ad}(G))$ for $A = C_*G$ has an A_∞-algebra structure arising from the A-coalgebra structure of $B(k,A,A)$ [10] and from the morphism $\tilde{\mu} : \Delta^*(\mathrm{Ad}\otimes\mathrm{Ad}) \to \mathrm{Ad}$ of A_∞-modules over A of Proposition 4.3.5. This A_∞-algebra structure should be equivalent to that of $CH^*(A,A)$ under the equivalences of Chapter 4.

5.3 Gerstenhaber and BV Structures

5.3.1 Relating the Hochschild and Ext/Tor cap products

We introduce the notion of a cap-pairing between simplicial and cosimplicial spaces (or spectra), modeled on the cap product of Hochschild cochains on chains, in analogy with the cup-pairing of McClure and Smith.

Definition 5.3.1 Let $X^\bullet$ be a cosimplicial space, and let $Y_\bullet$ and $Z_\bullet$ be simplicial spaces. A *cap-pairing* $c : (X^\bullet, Y_\bullet) \to Z_\bullet$ is a family of maps

$$c_{p,q} : X^p \times Y_{p+q} \to Z_q$$

satisfying the following relations:

$$\begin{aligned}
c_{p,q}(d^i f, x) &= c_{p-1,q}(f, d_i x), && 0 \le i \le p,\\
c_{p,q}(d^p f, x) &= d_0 c_{p-1,q+1}(f, x), &&\\
c_{p-1,q}(f, d_{p+i} x) &= d_{i+1} c_{p-1,q+1}(f, x), && 0 \le i < q,\\
c_{p,q}(s^i f, x) &= c_{p+1,q}(f, s_i x), && 0 \le i \le p,\\
c_{p,q}(f, s_{p+i} x) &= s_i c_{p+1,q-1}(f, x), && 0 \le i < q.
\end{aligned}$$

Note all of these relations hold in Z_q.

A *morphism* of cap-pairings from $c : (X^\bullet, Y_\bullet) \to Z_\bullet$ to $c' : (X'^\bullet, Y'_\bullet) \to Z'_\bullet$ is a triple consisting of a cosimplicial map $\mu_1 : X \to X'$ and simplicial maps $\mu_2 : Y \to Y'$ and $\mu_3 : Z \to Z'$ such that

$$\mu_3 \circ c_{p,q} = c'_{p,q} \circ (\mu_1 \times \mu_2) \quad \text{for all } p, q.$$

Analogous constructions pertain to simplicial and cosimplicial spectra. ∎

Just as a cup-pairing $\phi : (X^\bullet, Y^\bullet) \to Z^\bullet$ induces a family of maps $\operatorname{Tot} X^\bullet \times \operatorname{Tot} Y^\bullet \to \operatorname{Tot} Z^\bullet$, a cap-pairing induces a map $\operatorname{Tot} X^\bullet \times |Y_\bullet| \to |Z_\bullet|$:

Proposition 5.3.2 Let $c : (X^\bullet, Y_\bullet) \to Z_\bullet$ be a cap-pairing. Then for each u with $0 < u < 1$, c induces a map

$$\bar{c}_u : \operatorname{Tot} X^\bullet \times |Y_\bullet| \to |Z_\bullet|.$$

A morphism $(\mu_1, \mu_2, \mu_3) : c \to c'$ of cap-pairings induces a commuting diagram

$$\begin{array}{ccc} \operatorname{Tot} X^\bullet \times |Y_\bullet| & \xrightarrow{\bar{c}_u} & |Z_\bullet| \\ {\scriptstyle \operatorname{Tot}\mu_1 \times |\mu_2|} \downarrow & & \downarrow {\scriptstyle |\mu_3|} \\ \operatorname{Tot} X'^\bullet \times |Y'_\bullet| & \xrightarrow{\bar{c}'_u} & |Z'_\bullet| \end{array}$$

Proof: This follows from the same prismatic subdivision techniques used [32] to produce the maps $\bar{\phi}_u : \operatorname{Tot} X^\bullet \times \operatorname{Tot} Y^\bullet \to \operatorname{Tot} Z^\bullet$ from a cup-pairing $\phi : (X^\bullet, Y^\bullet) \to Z^\bullet$. For $n \geq 0$, define

$$D^n = \left(\coprod_{p=0}^{n} \Delta^p \times \Delta^{n-p} \right) / \sim,$$

where $\sim$ denotes the identifications $(d^{p+1}s, t) \sim (s, d^0 t)$ for $s \in \Delta^p$ and $t \in \Delta^{n-p-1}$. For each u, let $\sigma^n(u) : D^n \to \Delta^n$ be defined on $(s,t) \in \Delta^p \times \Delta^{n-p}$ by

$$\sigma^n(u)(s,t) = (us_0, \ldots, us_{p-1}, us_p + (1-u)t_0, (1-u)t_1, \ldots, (1-u)t_{n-p}).$$

Then for $0 < u < 1$, $\sigma^n(u)$ is a homeomorphism. We use $\sigma^n(u)$ to define the map $\bar{c}_u$. Take $f \in \operatorname{Tot} X^\bullet$ and $(s,y) \in |Y_\bullet|$, and recall that f is a sequence $(f_0, f_1, \ldots)$ of functions $f_n : \Delta^n \to X^n$ commuting with the cosimplicial structure maps of $\Delta^\bullet$ and $X^\bullet$. Suppose $s \in \Delta^{p+q}$ and $y \in Y_{p+q}$, and that $\sigma^n(u)^{-1}(s) = (s', s'') \in \Delta^p \times \Delta^q \subset D^{p+q}$. Then

$$\bar{c}_u(f, (s,y)) = (s'', c_{p,q}(f(s'), y)) \in \Delta^q \times Z_q.$$

The properties in the definition of the cap-pairing ensure that this map is well-defined: the second face-coface relation shows that this map is well-defined if a different representative is taken for $\sigma^n(u)^{-1}(s)$, and the other relations show that the map is well-defined for different representatives of $(s, y) \in |Y_\bullet|$.

The naturality of these constructions in the simplicial and cosimplicial objects then shows that a morphism of cap-pairings induces such a commuting diagram. ∎

We now apply this cap-pairing framework to the simplicial and cosimplicial spectra above.

Proposition 5.3.3 $S[G^c] \wedge_G EG_+$ is a right module for the ring spectrum $F_G(EG_+, S[G^c])$. Under the equivalences above, this module structure is equivalent to the $THH_S(S[G])$-module structure of $THH^S(S[G])$.

Proof: We first explain the module structure of $S[G^c] \wedge_G EG_+$ in terms of a cap-pairing between cosimplicial and simplicial spectra. Recall that $F_G(EG_+, S[G^c]) = \operatorname{Tot} Z^\bullet$, where $Z^n = \operatorname{Map}_G(B_n(G, G, *), S[G^c])$, and that $S[G^c] \wedge_G EG_+ \simeq |W_\bullet|$, where $W_n = S[G^c] \wedge_G B_n(G, G, *)_+$. The cap-pairing is then a collection of compatible maps

$$c_{p,q} : Z^p \wedge W_{p+q} \to W_q$$

given on elements $a \in Z^p$ and $s \wedge g[g_1 \mid \dots \mid g_{p+q}] \in W_{p+q}$ by

$$c_{p,q}(a, (s \wedge g[g_1 \mid \dots \mid g_{p+q}])) = sa(g[g_1 \mid \dots \mid g_p]) \wedge gg_1 \cdots g_p[g_{p+1} \mid \dots \mid g_{p+q}].$$

As with the cup-pairing on $Z^\bullet$, this map comes from the simplicial diagonal on $B_*(G, G, *)$ composed with the Alexander-Whitney approximation, and then applying the map to the left factor of the diagonal.

We show that this cap-pairing is compatible with the cup-pairing on $Z^\bullet$ giving rise to the ring structure on $F_G(EG_+, S[G^c])$, and in fact makes $S[G^c] \wedge_G EG_+$ a right $F_G(EG_+, S[G^c])$-module. Take $a \in Z^p$, $b \in Z^q$, and $c = s \wedge g[g_1 \mid \dots \mid g_{p+q+r}] \in W_{p+q+r}$. Then

$$\begin{aligned} c_{q,r}(b, c_{p,q+r}(a,c)) &= c_{q,r}(b, sa(g[g_1 \mid \dots \mid g_p]) \wedge gg_1 \cdots g_p[g_{p+1} \mid \dots \mid g_{p+q+r}]) \\ &= sa(g[g_1 \mid \dots \mid g_p])b(gg_1 \cdots g_p[g_{p+1} \mid \dots \mid g_{p+q}]) \\ &\qquad \wedge gg_1 \cdots g_{p+q}[g_{p+q+1} \mid \dots \mid g_{p+q+r}]) \\ &= s(a \cup b)(g[g_1 \mid \dots \mid g_{p+q}]) \wedge gg_1 \cdots g_{p+q}[g_{p+q+1} \mid \dots \mid g_{p+q+r}] \\ &= c_{p+q,r}(a \cup b, c) \end{aligned}$$

Similarly, the right $THH_S(S[G])$-module structure of $THH^S(S[G])$ via the Hochschild cap product can be described in terms of these cap pairings. As above, we have that Tot $Y^\bullet \simeq THH_S(S[G])$, where $Y^n = \mathrm{Map}(G^n, S[G])$, and that $|V_n| \simeq THH^S(S[G])$, where $V_n = S[G] \wedge_{G \times G^{\mathrm{op}}} B_n(G,G,G)_+$. Levelwise, $V_n = S[G] \wedge (G^n)_+$. Then there is a cap-pairing

$$h_{p,q} : Y^p \wedge V_{p+q} \to V_q$$

given by

$$h_{p,q}(f, a \wedge [g_1 \mid \dots \mid g_{p+q}]) = af([g_1 \mid \cdots \mid g_p]) \wedge [g_{p+1} \mid \dots \mid g_{p+q}].$$

This cap-pairing thus comes from evaluating the p-cochain on the first p factors of the $(p+q)$-chain. A simple calculation shows that this cap-pairing is compatible with the cup-pairing on $Y^\bullet$ and therefore induces the desired right $THH_S(S[G])$-module structure on $THH^S(S[G])$.

We now show that the isomorphisms of simplicial and cosimplicial spectra $\psi^\bullet : Z^\bullet \to Y^\bullet$ and $\chi_\bullet : W_\bullet \to V_\bullet$ are compatible with the cap-pairings c and h. Hence, we check that

$h_{p,q} \circ (\psi^p \wedge \chi_{p+q}) = \chi_q c_{p,q}$:

$$\begin{aligned} h_{p,q}(\psi^p(f) \wedge \chi_{p+q}(a \wedge [g_1 \mid \cdots \mid g_{p+q}])) \\ &= (g_1 \cdots g_{p+q})^{-1} a \psi^p(f)([g_1 \mid \cdots \mid g_p]) \wedge [g_{p+1} \mid \cdots \mid g_{p+q}] \\ &= (g_1 \cdots g_{p+q})^{-1} a f([g_1 \mid \cdots \mid g_p]) g_1 \cdots g_p \wedge [g_{p+1} \mid \cdots \mid g_{p+q}] \\ &= \chi_q((g_1 \cdots g_p)^{-1} a f([g_1 \mid \cdots \mid g_p]) g_1 \cdots g_p \wedge [g_{p+1} \mid \cdots \mid g_{p+q}]) \\ &= \chi_q(a f([g_1 \mid \cdots \mid g_p]) \wedge g_1 \cdots g_p [g_{p+1} \mid \cdots \mid g_{p+q}]) \\ &= \chi_q(c_{p,q}(f \wedge (a \wedge [g_1 \mid \cdots \mid g_{p+q}]))). \end{aligned}$$

Since this holds, the right action of $F_G(EG_+, S[G^c])$ on $S[G^c] \wedge_G EG_+$ is equivalent to that of $THH_S(S[G])$ on $THH^S(S[G])$. ∎

Under the equivalences of Section 5.2.2, these module structures should be equivalent to Klein's module structure of $S[LM]$ over the A_∞-ring spectrum LM^{-TM} [27].

Again applying $- \wedge Hk$ and passing to the derived category of chain complexes, we obtain that $\mathrm{Ad}(\Omega M) \otimes^L_{C_* \Omega M} k$ is a right A_∞-module for the A_∞-algebra $R\,\mathrm{Hom}_{C_* \Omega M}(k, \mathrm{Ad}(\Omega M))$, and that this module structure is equivalent to that of the Hochschild cochains acting on the Hochschild chains.

We now relate these cap products to the evaluation map and to the isomorphism D.

Proposition 5.3.4 View $\eta : k \to \mathrm{Ad}(\Omega M)$ as a map of $C_* \Omega M$-modules, inducing a map $\eta_* : \mathrm{Tor}_*^{C_* \Omega M}(k, k) \to \mathrm{Tor}_*^{C_* \Omega M}(\mathrm{Ad}, k)$. Then for $z \in \mathrm{Tor}_*^{C_* \Omega M}(k, k)$, $f \in \mathrm{Ext}^*_{C_* \Omega M}(k, \mathrm{Ad}(\Omega M))$,

$$\mathrm{ev}_z(f) = (-1)^{|f||z|} \eta_*(z) \cap f.$$

Proof: By the form of the cap-pairing on the spectrum level, the cap product

$$\mathrm{Ext}^*_{C_* \Omega M}(k, \mathrm{Ad}(\Omega M)) \otimes \mathrm{Tor}_*^{C_* \Omega M}(\mathrm{Ad}(\Omega M), k) \to \mathrm{Tor}_*^{C_* \Omega M}(\mathrm{Ad}(\Omega M), k)$$

is given by the sequence of maps

$$\begin{aligned}\mathrm{Ext}^*_{C_*\Omega M}(k,\mathrm{Ad})\otimes\mathrm{Tor}^{C_*\Omega M}_*(\mathrm{Ad},k) &\xrightarrow{\cong}\mathrm{Ext}^*_{C_*\Omega M}(k,\mathrm{Ad})\otimes\mathrm{Tor}^{C_*\Omega M}_*(\mathrm{Ad},\Delta^*(k\otimes k))\\ &\xrightarrow{\mathrm{ev}}\mathrm{Tor}^{C_*\Omega M}_*(\mathrm{Ad},\Delta^*(\mathrm{Ad}\otimes k))\xrightarrow{\cong}\mathrm{Tor}^{C_*\Omega M}_*(\Delta^*(\mathrm{Ad}\otimes\mathrm{Ad}),k)\xrightarrow{\tilde\mu_*}\mathrm{Tor}^{C_*\Omega M}_*(\mathrm{Ad},k),\end{aligned}$$

where $\tilde\mu$ is the morphism of A_∞-modules over $C_*\Omega M$ given in Prop. 4.3.5. By introducing an extra k factor via Δ_k and then collapsing it via λ, $\mathrm{ev}:\mathrm{Ext}^*_{C_*\Omega M}(k,\mathrm{Ad})\otimes\mathrm{Tor}^{C_*\Omega M}_*(k,k)\to\mathrm{Tor}^{C_*\Omega M}_*(\mathrm{Ad},k)$ is similarly given by

$$\begin{aligned}\mathrm{Ext}^*_{C_*\Omega M}(k,\mathrm{Ad})\otimes\mathrm{Tor}^{C_*\Omega M}_*(k,k) &\xrightarrow{\cong}\mathrm{Ext}^*_{C_*\Omega M}(k,\mathrm{Ad})\otimes\mathrm{Tor}^{C_*\Omega M}_*(k,\Delta^*(k\otimes k))\\ &\xrightarrow{\mathrm{ev}}\mathrm{Tor}^{C_*\Omega M}_*(k,\Delta^*(\mathrm{Ad}\otimes k))\xrightarrow{\cong}\mathrm{Tor}^{C_*\Omega M}_*(\Delta^*(k\otimes\mathrm{Ad}),k)\xrightarrow{\lambda_*}\mathrm{Tor}^{C_*\Omega M}_*(\mathrm{Ad},k)\end{aligned}$$

Then for a given $f\in\mathrm{Ext}^*_{C_*\Omega M}(k,\mathrm{Ad}(\Omega M))$, the sequence of squares

$$\begin{array}{ccccccc}\mathrm{Tor}(k,k) & \longrightarrow & \mathrm{Tor}(k,\Delta^*(k\otimes k)) & \xrightarrow{\mathrm{ev}(f)} & \mathrm{Tor}(k,\Delta^*(\mathrm{Ad}\otimes k)) & \longrightarrow & \mathrm{Tor}(\Delta^*(k\otimes\mathrm{Ad}),k)\\ \downarrow{\scriptstyle\eta_*} & & \downarrow{\scriptstyle\eta_*} & & \downarrow{\scriptstyle\eta_*} & & \downarrow{\scriptstyle\eta_*}\\ \mathrm{Tor}(\mathrm{Ad},k) & \to & \mathrm{Tor}(\mathrm{Ad},\Delta^*(k\otimes k)) & \xrightarrow{\mathrm{ev}(f)} & \mathrm{Tor}(\mathrm{Ad},\Delta^*(\mathrm{Ad}\otimes k)) & \to & \mathrm{Tor}(\Delta^*(\mathrm{Ad}\otimes\mathrm{Ad}),k)\end{array}$$

commutes. Since μ is unital, $\mu(\eta\circ\mathrm{Ad})=\lambda:\Delta^*(k\otimes\mathrm{Ad})\to\mathrm{Ad}$ as maps of chain complexes. Hence, applying $\mathrm{Tor}^{C_*\Omega M}_*(-,k)$ to the composite

$$\Delta^*(k\otimes\mathrm{Ad})\xrightarrow{\eta\otimes\mathrm{Ad}}\Delta^*(\mathrm{Ad}\otimes\mathrm{Ad})\xrightarrow{\tilde\mu}\mathrm{Ad}$$

gives $\mathrm{Tor}^{C_*\Omega M}_*(\lambda,k)$. Taking into account the swap between the Ext and Tor tensor factors for the cap product, this establishes the identity $\mathrm{ev}_z(f)=(-1)^{|f||z|}\eta_*(z)\cap f$. ∎

Proposition 5.3.5 The isomorphism $D:HH^*(C_*\Omega M)\to HH_{*+d}(C_*\Omega M)$ is given by

$$D(f)=(-1)^{|f|d}z_H\cap f,$$

where $z_H \in HH_d(C_*\Omega M)$ is the image of $[M] \in \mathrm{Tor}_d^{C_*\Omega M}(k,k)$ under the maps

$$\mathrm{Tor}_*^{C_*\Omega M}(k,k) \xrightarrow{\mathrm{Tor}(\eta,k)} \mathrm{Tor}_*^{C_*\Omega M}(\mathrm{Ad}(\Omega M),k) \xrightarrow{\Lambda_*^{-1}} HH_*(C_*\Omega M).$$

Proof: By construction, $D(f) = \Lambda_*^{-1}(\mathrm{ev}_{[M]}(\Lambda^* f))$. By the above proposition,

$$\begin{aligned} \Lambda_*^{-1}(\mathrm{ev}_{[M]}(\Lambda^* f)) &= (-1)^{|f|d}\Lambda_*^{-1}(\eta_*[M] \cap \Lambda^* f) \\ &= (-1)^{|f|d}\Lambda_*^{-1}(\eta_*[M]) \cap f = (-1)^{|f|d} z_H \cap f. \end{aligned}$$

■

Proposition 5.3.6 $B(z_H) = 0$.

Proof: Observe that we have the following commutative diagram:

$$\begin{array}{ccccc} H_*(M) & \xrightarrow{\cong} & HH_*(C_*\Omega M, k) & \xrightarrow[\cong]{\Lambda_*} & \mathrm{Tor}_*^{C_*\Omega M}(k,k) \\ \downarrow{\scriptstyle c_*} & & & & \downarrow{\scriptstyle \mathrm{Tor}(\eta,k)} \\ H_*(LM) & \xrightarrow[\cong]{BFG} & HH_*(C_*\Omega M, C_*\Omega M) & \xrightarrow[\cong]{\Lambda_*} & \mathrm{Tor}_*^{C_*\Omega M}(\mathrm{Ad}(\Omega M),k) \end{array}$$

where $c : M \to LM$ is the map sending $x \in M$ to the constant loop at x. Then $B(z_H) = BFG(\Delta(c_*[M]))$. The trivial action of S^1 on M induces a degree-1 operator Δ on $H_*(M)$ that is identically 0. Since c is S^1-equivariant with respect to these actions, $\Delta \circ c_* = c_* \circ \Delta = 0$, so $B(z_H) = 0$.

■

5.3.2 The BV structures on $HH^*(C_*\Omega M)$ and String Topology

Now that we have shown that D arises as a cap product in Hochschild homology, we may employ an algebraic argument of Ginzburg [16], with sign corrections by Menichi [34], to show that this gives $HH^*(C_*\Omega M)$ the structure of a BV algebra.

For any DGA A, the cup product on $HH^*(A)$ is graded-commutative, so the right cap-product action of $HH^*(A)$ on $HH_*(A)$ also defines a left action, with $a \cdot z = (-1)^{|a||z|} z \cap a$

for $z \in HH_*(A)$ and $a \in HH^*(A)$. Hence, each $a \in HH^*(A)$ defines a degree-$|a|$ operator i_a on $HH_*(A)$ by $i_a(z) = a \cdot z$. Then $D(a) = a \cdot z_H = i_a(z_H)$.

Similarly, for each $a \in HH^*(A)$, there is a "Lie derivative" operator L_a on $HH_*(A)$ of degree $|a| + 1$, and there is the Connes B operator of degree 1. It is well known that these operations make $(HH^*(A), HH_*(A))$ into a calculus, an algebraic model of the interaction of differential forms and polyvector fields on a manifold. Tamarkin and Tsygan [43] in fact extend this calculus structure to a notion of ∞-calculus on the Hochschild chains and cohains of A, which descends to the usual calculus structure on homology, and they provide explicit descriptions of the operations on the chain level. The Lie derivative in this calculus structure is the graded commutator

$$L_a = [B, i_a],$$

which for $a, b \in HH^*(A)$ satisfies the relations

$$i_{[a,b]} = (-1)^{|a|+1}[L_a, i_b] \quad \text{and} \quad L_{a \cup b} = L_a i_b + (-1)^{|a|} i_a L_b,$$

where $[a, b]$ is the usual Gerstenhaber Lie bracket in $HH^*(A)$.

Theorem 5.3.7 $HH^*(C_*\Omega M)$ is a BV algebra under the Hochschild cup product and the operator $\kappa = -D^{-1}BD$. The Lie bracket induced by this BV algebra structure is the standard Gerstenhaber Lie bracket.

Proof: Recall that $D(a) = a \cdot z_H$, so $B(a \cdot z_H) = -\kappa(a) \cdot z_H$. Then

$$\begin{aligned}
D([a, b]) &= i_{[a,b]}(z_H) \\
&= (-1)^{|a|+1}(L_a i_b - (-1)^{(|a|-1)|b|} i_b L_a)(z_H) \\
&= (-1)^{|a|+1}(B i_a i_b - (-1)^{|a|} i_a B i_b - (-1)^{(|a|-1)|b|} i_b B i_a + (-1)^{(|a|-1)|b|+|a|} i_b i_a B)(z_H) \\
&= (-1)^{|a|+1} B((a \cup b) \cdot z_H) + a \cdot B(b \cdot z_H) + (-1)^{|a||b|+|b|+|a|} b \cdot B(a \cdot z_H) \\
&= (-1)^{|a|} \kappa(a \cup b) \cdot z_H - (a \cup \kappa(b)) \cdot z_H - (-1)^{|a||b|+|b|+|a|}(b \cup \kappa(a)) \cdot z_H \\
&= ((-1)^{|a|} \kappa(a \cup b) - (-1)^{|a|} \kappa(a) \cup b - a \cup \kappa(b)) \cdot z_H
\end{aligned}$$

so therefore

$$[a, b] = (-1)^{|a|} \kappa(a \cup b) - (-1)^{|a|} \kappa(a) \cup b - a \cup \kappa(b).$$

Since $HH^*(C_*\Omega M)$ is a Gerstenhaber algebra under $\cup$ and $[\ ,\ ]$, this identity shows that it is a BV algebra under $\cup$ and κ. ∎

Theorem 5.3.8 Under the isomorphism $BFG \circ D : HH^*(C_*\Omega M) \to H_{*+d}(LM)$, the BV algebra structure above coincides with the BV algebra structure of string topology.

Proof: We have seen that the isomorphism $HH^*(C_*\Omega M) \cong H_{*+d}(LM)$ coming from spectra coincides with the composite isomorphism $BFG \circ D$, and so the latter takes the Hochschild cup product to the Chas-Sullivan loop product. Furthermore,

$$BFG \circ D \circ \kappa = -BFG \circ B \circ D = -\Delta \circ BFG \circ D,$$

so $BFG \circ D$ takes κ to $-\Delta$, the negative of the BV operator on string topology.

Tamanoi gives an explicit homotopy-theoretic construction of the loop bracket and BV operator in string topology [42]. In his Section 5, he notes that the bracket associated to the usual Δ operator is actually the negative $-\{-,-\}$ of the loop bracket, as defined using Thom spectrum constructions. Consequently, $-\Delta$ should be the correct BV operator on $\mathbb{H}_*(LM)$, since the sign change carries through to give $\{-,-\}$ as the bracket induced from the BV algebra structure. Then the Hochschild Lie bracket $[-,-]$ does correspond to the loop bracket under this isomorphism.

We conclude that $BFG \circ D$ is an isomorphism of BV algebras from $(HH^*(C_*\Omega M), \cup, \kappa)$ to the string topology BV algebra $(H_{*+d}(LM), \circ, -\Delta)$. ∎

We also compare this result to the previous BV algebra isomorphisms between string topology and Hochschild homology. We note that Vaintrob's argument in [46] relies on Ginzburg's algebraic argument without Menichi's sign corrections. With those sign changes in place, the argument appears to carry through to produce $-D^{-1}BD$ as the appropriate BV operator on Hochschild cohomology, and thus to give $-\Delta$ as the BV operator on string topology.

As noted above, Felix and Thomas also construct a BV algebra isomorphism between $\mathbb{H}_*(LM)$ and $HH^*(C^*M)$ when M is simply connected and when k is a field of characteristic 0 [13]. They invoke results of Menichi on cyclic cohomology [33] and of Tradler and

Zeinalian [44, 45] to state that their BV operator on $HH^*(C^*M)$ induces the Gerstenhaber Lie bracket. In light of the sign change above and the isomorphism of Gerstenhaber algebras $HH^*(C^*M) \cong HH^*(C_*\Omega M)$ of Felix, Menichi, and Thomas for M simply connected, it would be of interest to trace through these isomorphisms to check the sign of the induced bracket in their context.

Appendix A

Algebraic Structures

A.1 Chain Complexes and Differential Graded Algebra

A.1.1 Chain Complexes

Recall that k denotes a fixed commutative ring.

Definition A.1.1 Let $\mathrm{Ch}(k)$ denote the category of unbounded chain complexes of k-modules, with differential of degree -1, where the morphisms are chain maps of complexes. Given a homogeneous element a of a complex, let $|a|$ denote its degree. Given chain complexes A, B, define their *tensor product* $A \otimes_k B$ by

$$(A \otimes_k B)_n = \bigoplus_{j \in \mathbb{Z}} A_j \otimes_k B_{n-j},$$

with differential $d_{A\otimes B} = d_A \otimes \mathrm{id}_B + \mathrm{id}_A \otimes d_B$, and define the complex $\mathrm{Hom}_k(A, B)$ of *k-linear maps* from A to B by

$$\mathrm{Hom}_k(A, B)_n = \prod_{j \in \mathbb{Z}} \mathrm{Hom}_k(A_j, B_{j+n}),$$

with differential $Df = d_B f - (-1)^{|f|} f d_A$. Note that the chain maps $M \to N$ are precisely the 0-cycles in $\mathrm{Hom}_k(M, N)$, and that $f, g \in \mathrm{Hom}_k(M, N)$ are homotopic if and only if $f - g = Dh$ for some h. When k is clear from context, we write $\otimes$ for $\otimes_k$ and Hom for Hom_k. ∎

We follow the Koszul convention that if two homogeneous elements a, b of some chain complex are transposed, we introduce a factor $(-1)^{|a||b|}$. Thus, since $|d| = -1$, $d(a \otimes b) = (d \otimes \mathrm{id} + \mathrm{id} \otimes d)(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes db$. Note that the differential is written only as d when the complex is clear from context.

Let k also denote the chain complex with k in degree 0 and 0 elsewhere. Then there are obvious isomorphisms $\lambda_A : k \otimes A \to A$ and $\rho_A : A \otimes k \to A$ for each chain complex A, so k is a unit for $\otimes$.

Definition A.1.2 For $n \in \mathbb{Z}$, let S^n be the chain complex consisting of k in degree n and 0 elsewhere. Given $A \in \mathrm{Ch}(k)$, let the *suspension* ΣA of A be the complex $A \otimes S^1$, with the differential arising from the tensor product. ▪

Under the natural identifications $S^n \cong S^{1^{\otimes n}}$, $\Sigma^n A \cong A \otimes S^n$. This suspension construction is of importance in the construction of a cofibrantly generated model structure on $\mathrm{Ch}(k)$ and related categories, as discussed in Sections 2.2 and A.2.

Definition A.1.3 For A, B chain complexes, define the algebraic twist map $\tau_{A,B} : A \otimes B \to B \otimes A$ by $\tau_{A,B}(a \otimes b) = (-1)^{|a||b|} b \otimes a$. If A, B are clear from context, $\tau_{A,B}$ is written τ. ▪

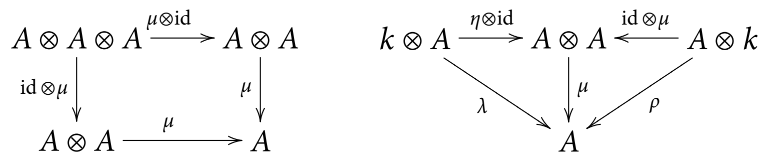
Then $(\mathrm{Ch}(k), \otimes, k)$ with the τ morphisms and the internal Hom-objects above is a closed symmetric monoidal category (see [29, §VII.7]). The notation below based on the symmetric group simplifies the process of manipulating composites of τ morphisms.

Notation A.1.4 Suppose that $\sigma \in S_n$ is a permutation on n letters $\{1, \ldots, n\}$. Denote by $\tau_{n,\sigma}$, or τ_σ if n is understood, the unique morphism $X_1 \otimes \cdots \otimes X_n \to X_{\sigma^{-1}(1)} \otimes \cdots \otimes X_{\sigma^{-1}(n)}$ composed of the τ_{X_i,X_j} and taking the ith factor in the source to the $\sigma(i)$th factor in the target. Consequently, for $\rho, \sigma \in S_n$, $\tau_\rho \circ \tau_\sigma = \tau_{\rho\sigma}$. ▪

A.1.2 Differential Graded Algebras, Coalgebras, and Hopf Algebras

Definition A.1.5 A *differential graded algebra* (or *DGA*) is a monoid in $\mathrm{Ch}(k)$, i.e., a chain complex $A \in \mathrm{Ch}(k)$ with a "multiplication" chain map $\mu : A \otimes A \to A$ and a "unit" chain map

$\eta : k \to A$ such that the associativity and unitality diagrams



all commute. If A is concentrated in nonnegative (resp., nonpositive) degrees, it is called a *chain algebra* (resp., *cochain algebra*). A *differential graded coalgebra* (or *DGC*) is a comonoid in $\mathrm{Ch}(k)$, i.e., a chain complex C with a coassociative comultiplication $\Delta : C \to C \otimes C$ and a counit $\epsilon : C \to k$.

A morphism $\phi : A \to B$ of DGAs is a chain map such that $\phi\mu_A = \mu_B(\phi \otimes \phi)$ and $\phi\eta_A = \eta_B$, and similarly for morphisms of DGCs. ∎

Notation A.1.6 We introduce a convention known as *Sweedler notation* [40]. Suppose C is a DGC, and take $c \in C$. We write the coproduct of c as $\Delta(c) = \sum_c c^{(1)} \otimes c^{(2)}$, with the index c indicating a sum over the relevant summands of $\Delta(c)$. By the coassociativity of Δ,

$$(\mathrm{id} \otimes \Delta)(\Delta(c)) = \sum_c c^{(1)} \otimes c^{(2,1)} \otimes c^{(2,2)} = \sum_c c^{(1,1)} \otimes c^{(1,2)} \otimes c^{(2)} = (\Delta \otimes \mathrm{id})(\Delta(c)).$$

We instead denote this twice-iterated coproduct unambiguously as $\Delta^2(c) = \sum_c c^{(1)} \otimes c^{(2)} \otimes c^{(3)}$. Higher iterates $\Delta^n(c)$ are denoted similarly, with components $c^{(1)}, \ldots, c^{(n+1)}$. ∎

Example A.1.7 In Sweedler notation, the counital condition $\mathrm{id} = \rho(\mathrm{id} \otimes \epsilon)\Delta = \lambda(\epsilon \otimes \mathrm{id})\Delta$ becomes

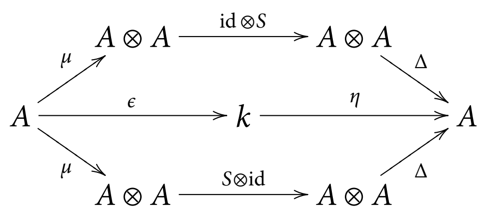
$$c = \sum_c c^{(1)} \epsilon(c^{(2)}) = \sum_c \epsilon(c^{(1)}) c^{(2)}.$$

∎

Proposition A.1.8 If A, B are DGAs, then $A \otimes B$ is a DGA with multiplication $(\mu_A \otimes \mu_B)\tau_{(23)}$ and unit $\lambda(\eta_A \otimes \eta_B)$. If C, D are DGCs, then $C \otimes D$ is a DGC with comultiplication $\tau_{(23)}(\Delta_C \otimes \Delta_D)$ and counit $(\epsilon_C \otimes \epsilon_D)\lambda^{-1}$. ∎

Note that $\tau_{A,B} : A \otimes B \to B \otimes A$ is an isomorphism of DGAs, and similarly $\tau_{C,D}$ is an isomorphism of DGCs.

Definition A.1.9 A k-module chain complex A is a *differential graded Hopf algebra* (or DGH) if it has maps $\mu : A \otimes A \to A$, $\eta : k \to A$, $\Delta : A \to A \otimes A$, $\epsilon : A \to k$ such that (A, μ, η) is a differential graded algebra, (A, Δ, ϵ) is a differential graded coalgebra, and Δ and ϵ are maps of DGAs (with the product DGA structure on $A \otimes A$). If A has a linear isomorphism $S : A \to A$ making the diagrams



commute, that is, $\Delta(\mathrm{id} \otimes S)\mu = \Delta(S \otimes \mathrm{id})\mu = \eta\epsilon$, then such an S is called an *antipode* for A.∎

A.1.3 Modules over a DGA

Suppose that A is a DGA. Recall that A is then a monoid in $\mathrm{Ch}(k)$, the category of chain complexes of k-modules.

Definition A.1.10 A left A-module is a chain complex M with a unital left action of A, i.e., a chain map $a : A \otimes M \to M$ such that $a(\mathrm{id} \otimes a) = a(\mu \otimes \mathrm{id})$ and $a(\eta \otimes \mathrm{id}) = \lambda$. Right A-modules are defined analogously. Morphisms of A-modules are chain maps compatible with the action of A. Denote the categories of left and right A-modules as A-Mod and Mod-A, respectively.

If B is another DGA, then an A-B-bimodule is a complex M with a left action of A and a right action of B that commute, and the category of such bimodules is denoted A-Mod-B. ∎

The observation that DGAs are precisely monoids in the closed symmetric monoidal category $\mathrm{Ch}(k)$ allows monoid-theoretic constructions of tensor products and complexes of A-linear maps for A-modules.

Definition A.1.11 Take $M, N \in A$-Mod and $P \in$ Mod-A. Note that $a_P \otimes \mathrm{id}_M$ and $\mathrm{id}_P \otimes a_M$ give two chain maps $P \otimes A \otimes M \to P \otimes M$, and define the tensor product $P \otimes_A M$ of P and M over A to be the cokernel of their difference.

Similarly, define chain maps $a_M^*, a_{N*} : \mathrm{Hom}(M,N) \to \mathrm{Hom}(A \otimes M, N)$ by $a_M^*(f) = f a_M$ and $a_{N*}(f) = a_N(\mathrm{id}_A \otimes f)$ for $f \in \mathrm{Hom}(M,N)$. Define the complex $\mathrm{Hom}_A(M,N)$ of A-linear maps to be $\ker(a_M^* - a_{N*})$. (A complex of A-linear maps for right A-modules is constructed similarly.) ▪

More concretely, let I be the subcomplex of $P \otimes M$ generated by $pa \otimes m - p \otimes am$, for all $p \in P, a \in A, m \in M$. Then $P \otimes_A M = P \otimes M/I$. Also, $f \in \mathrm{Hom}_A(M,N)$ is a k-linear map from M to N with $f(am) = (-1)^{|a||f|} a f(m)$ for all $a \in A, m \in M$.

Note also that if B and C are DGAs, then $\otimes_A$ gives a functor from B-Mod-$A \times A$-Mod-C to B-Mod-C, and $\mathrm{Hom}_A(-,-)$ gives functors from $(A\text{-Mod-}B)^{\mathrm{op}} \times A$-Mod-$C$ to B-Mod-C and from $(B\text{-Mod-}A)^{\mathrm{op}} \times C$-Mod-$A$ to C-Mod-B.

A.1.4 Pullbacks of Modules, Opposite Algebras, and Enveloping Algebras

Definition A.1.12 Suppose A, B are two DGAs and $f : A \to B$ is a morphism of DGAs. We define functors $f^* : B\text{-Mod} \to A\text{-Mod}$ and $f^{*\,\mathrm{op}} : \text{Mod-}B \to \text{Mod-}A$. For $M \in B$-Mod with action $a_M : B \otimes M \to M$, define $f^*M \in A$-Mod to be the chain complex M with A-action $a_M(f \otimes \mathrm{id})$. Similarly, for $N \in$ Mod-B with action a_N, define $f^{*\,\mathrm{op}}N \in$ Mod-A to be the chain complex N with A-action $a_N(\mathrm{id} \otimes f)$. The "op" notation is typically dropped when it is clear from context. These functors also apply to the appropriate categories of bimodules. ▪

This pullback construction is adjoint to related base-change functors for f that associate B-modules to A-modules:

Definition A.1.13 For $f : A \to B$ as above, define functors $f_!, f_* : A\text{-Mod} \to B\text{-Mod}$ by $f_!M = f^*B \otimes_A M$ and $f_*M = \mathrm{Hom}_A(f^*B, M)$. ▪

Proposition A.1.14 For $f : A \to B$ as above, $(f_!, f^*)$ and (f^*, f_*) are adjoint pairs of $Ch(k)$-enriched functors.

Proof: Note that $f^*M \cong f^*B \otimes_B M \cong \mathrm{Hom}_B(f^*B, M)$. By adjoint associativity,

$$\mathrm{Hom}_A(N, f^*M) \cong \mathrm{Hom}_A(N, \mathrm{Hom}_B(f^*B, M)) \cong \mathrm{Hom}_B(f^*B \otimes_A N, M) = \mathrm{Hom}_B(f_!N, M)$$
$$\mathrm{Hom}_A(f^*M, N) \cong \mathrm{Hom}_A(f^*B \otimes_B M, N) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(f^*B, N)) = \mathrm{Hom}_B(M, f_*N)$$

yielding the desired adjunctions. ■

Definition A.1.15 Let (A, μ, η) be a DGA. Define the *opposite algebra* $(A^{\mathrm{op}}, \mu^{\mathrm{op}}, \eta^{\mathrm{op}})$ by $A^{\mathrm{op}} = A$ as chain complexes, $\mu^{\mathrm{op}} = \mu\tau$ and $\eta^{\mathrm{op}} = \eta$. For a DGC (C, Δ, ϵ), define the *opposite coalgebra* $(C^{\mathrm{op}}, \Delta^{\mathrm{op}}, \epsilon^{\mathrm{op}})$ by $C^{\mathrm{op}} = C$, $\Delta^{\mathrm{op}} = \tau\Delta$, and $\epsilon^{\mathrm{op}} = \epsilon$.

Define the *enveloping algebra* A^e to be the product DGA $A \otimes A^{\mathrm{op}}$, and similarly for coalgebras. ■

Observe that $M \in A\text{-Mod}$ with action $a_M : A \otimes M \to M$ is a right A^{op}-module via $a_M\tau$. Similarly, right A-modules are equivalent to left A^{op}-modules. Hence. $M \in A\text{-Mod-}B$, with action $a_M : A \otimes M \otimes B \to M$ may be regarded as a left or right module in four distinct ways:

- M is a left $A \otimes B^{\mathrm{op}}$-module via the action $a_M\tau_{(23)}$,
- M is a left $B^{\mathrm{op}} \otimes A$-module via $a_M\tau_{(23)}\tau_{(12)} = a_M\tau_{(132)}$,
- M is a right $A^{\mathrm{op}} \otimes B$-module via $a_M\tau_{(12)}$,
- M is a right $B \otimes A^{\mathrm{op}}$-module via $a_M\tau_{(12)}(\mathrm{id} \otimes \tau_{(12)}) = a_M\tau_{(123)}$.

The second and fourth module structures are obtained by pullback along the DGA isomorphism τ. Note that when $B = A$, this characterizes A-A-bimodules as left and right modules over A^e. Call these module structures the *canonical module structures* associated to the A-B-bimodule M.

A.1.5 Hopf Algebras and Adjoint Actions

Suppose now that A is a DGH with an algebra anti-automorphism $S : A \to A$, so that $S : A \to A^{\mathrm{op}}$ is an isomorphism of DGAs. Also assume that $S^2 = \mathrm{id}$.

Definition A.1.16 For $\epsilon = 0, 1$, define $\mathrm{ad}_\epsilon = (\mathrm{id} \otimes S)\tau^\epsilon\Delta : A \to A \otimes A^{\mathrm{op}}$. Note that ad_0 and ad_1 are both DGA morphisms, since Δ and τ are. For $M \in A\text{-Mod-}A$, $\mathrm{ad}_\epsilon^* M$ is defined using the canonical A^e-module structure. ■

Suppose now that S is an antipode for A. Then the coinvariant module $k \otimes_A \mathrm{ad}^* A^e$ is isomorphic to A as A^e-modules.

Proposition A.1.17 If A is a DGH with antipode S, then $\phi : \operatorname{ad}_0^* A^e \otimes_A k \to A$ given by $(a \otimes a') \otimes \lambda \mapsto \lambda a a'$ is an isomorphism of left A^e-modules, and $\phi' : k \otimes_A \operatorname{ad}_1^* A^e \to A$ given by $\lambda \otimes (a \otimes a') = (-1)^{|a||a'|} \lambda a' a$ is an isomorphism of right A^e-modules. ∎

In fact, these isomorphisms are induced from isomorphisms of A^e-resolutions for these modules, which we state below.

Proposition A.1.18 Let A be a DGH with antipode S. There are simplicial isomorphisms

$$\gamma_\bullet^{L,0} : B_\bullet(A, A, A) \leftrightarrows B_\bullet(\operatorname{ad}_0^* A^e, A, k) : \phi_n^{L,0}$$
$$\gamma_\bullet^{R,1} : B_\bullet(A, A, A) \leftrightarrows B_\bullet(k, A, \operatorname{ad}_1^* A^e) : \phi_n^{R,1}$$

which descend to isomorphisms on the corresponding realizations. When $S^2 = \mathrm{id}$, there are isomorphisms $(\gamma_\bullet^{L,1}, \phi_\bullet^{L,1})$ and $(\gamma_\bullet^{R,0}, \phi_\bullet^{R,0})$ in the opposite ϵ-cases as well.

Proof: We first exhibit an isomorphism $\gamma_\bullet^{L,0} : B_\bullet(A, A, A) \to B_\bullet(\operatorname{ad}_0^* A^e, A, k)$ and its inverse:

$$\gamma_n^{L,0}(a[a_1 \mid \cdots \mid a_n]a') = \pm(a \otimes (a_1 \cdots a_n)^{(2)} a')[a_1^{(1)} \mid \cdots \mid a_n^{(1)}]$$
$$\phi_n^{L,0}((b \otimes b')[b_1 \mid \cdots \mid b_n]) = \pm b[b_1^{(1)} \mid \cdots \mid b_n^{(1)}] S((b_1 \cdots b_n)^{(2)}) b'.$$

It is straightforward to verify that these are isomorphisms of simplicial A^e-modules and thus determine isomorphisms of the associated bar complexes.

Next, we show an isomorphism $\gamma_\bullet^{R,1} : B_\bullet(A, A, A) \to B_\bullet(k, A, \operatorname{ad}_1^* A^e)$ and its inverse:

$$\gamma_n^{R,1}(a'[a_1 \mid \cdots \mid a_n]a) = \pm[a_1^{(2)} \mid \cdots \mid a_n^{(2)}](a \otimes a' S((a_1 \cdots a_n)^{(1)}))$$
$$\phi_n^{R,1}([b_1 \mid \cdots \mid b_n](b \otimes b')) = \pm b'(b_1 \cdots b_n)^{(1)}[b_1^{(2)} \mid \cdots \mid b_n^{(2)}]b$$

When $S^2 = \mathrm{id}$, we have the isomorphisms

$$\gamma_n^{L,1}(a[a_1 \mid \cdots \mid a_n]a') = \pm(a \otimes (a_1 \cdots a_n)^{(1)} a')[a_1^{(2)} \mid \cdots \mid a_n^{(2)}]$$
$$\phi_n^{L,1}((b \otimes b')[b_1 \mid \cdots \mid b_n]) = \pm b[b_1^{(2)} \mid \cdots \mid b_n^{(2)}] S((b_1 \cdots b_n)^{(1)}) b',$$

which assemble to an isomorphism $\gamma_\bullet^{L,1} : B_\bullet(A, A, A) \to B_\bullet(\mathrm{ad}_1^* A^e, A, k)$, and isomorphisms

$$\gamma_n^{R,0}(a'[a_1 \mid \cdots \mid a_n]a) = \pm[a_1^{(1)} \mid \cdots \mid a_n^{(1)}](a \otimes a' S((a_1 \cdots a_n)^{(2)}))$$
$$\phi_n^{R,0}([b_1 \mid \cdots \mid b_n](b \otimes b')) = \pm b'(b_1 \cdots b_n)^{(2)}[b_1^{(1)} \mid \cdots \mid b_n^{(1)}]b,$$

which produce an isomorphism $\gamma_\bullet^{R,0} : B_\bullet(A, A, A) \to B_\bullet(k, A, \mathrm{ad}_0^* A^e)$. ∎

Relating these simplicial isomorphisms back to Prop. A.1.17, note that, for example, that $\mathrm{ad}_0^* A^e \otimes_A k$ is the cokernel of $d_0 - d_1 : B_1(\mathrm{ad}_0^* A^e, A, k) \to B_0(\mathrm{ad}_0^* A^e, A, k)$, and that A is the cokernel of $d_0 - d_1 : B_1(A, A, A) \to B_0(A, A, A)$. Hence, the simplicial isomorphisms induces isomorphisms on these cokernels.

In fact, these simplicial isomorphisms hold for a Hopf object H in an arbitrary symmetric monoidal category $(\mathcal{C}, \otimes, I)$, with a monoid anti-automorphism $S : H \to H$. For example, we apply this result to a topological group G considered as a Hopf object in the category Top in Proposition 4.2.7.

A.1.6 Gerstenhaber and Batalin-Vilkovisky Algebras

We recall from [15] the standard definitions of a Gerstenhaber algebra and a Batalin-Vilkovisky algebra.

Definition A.1.19 A *Lie bracket of degree m* on a graded k-module V is a Lie bracket on $\Sigma^m V$, that is, a bilinear map $[-,-] : V \otimes V \to V$ satisfying graded anti-commutativity

$$[u, v] = -(-1)^{(|u|-m)(|v|-m)}[v, u]$$

and the graded Jacobi identity

$$[u, [v, w]] = [[u, v], w] + (-1)^{(|u|-m)(|v|-m)}[v, [u, w]]$$

on homogeneous elements $u, v, w \in V$.

A *Gerstenhaber algebra* is a graded k-module A together with a graded-commutative multiplication and a degree-1 Lie bracket that are compatible via the Poisson relation

$$[a, bc] = [a, b]c + (-1)^{|b|(|a|-1)} b[a, c].$$

on homogeneous elements $a, b, c \in A$,

A *Batalin-Vilkovisky (BV) algebra* is a graded k-module A together with a graded-commutative multiplication and a degree-1 operator Δ with $\Delta^2 = 0$, so that Δ is a differential operator of order 2 on A:

$$\Delta(abc) = \Delta(ab)c + (-1)^{|a|} a\Delta(bc) + (-1)^{(|a|-1)|b|} b\Delta(ac) - (\Delta a)bc - (-1)^{|a|} a(\Delta b)c - (-1)^{|a|+|b|} ab(\Delta c). \quad \blacksquare$$

Getzler shows algebraically that a BV algebra has a canonically defined Gerstenhaber algebra structure, with the bracket given by

$$[a, b] = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}(\Delta a)b - a(\Delta b).$$

Conversely, if Δ is such that this induced bracket is a Gerstenhaber algebra, then it makes A a BV algebra.

These two structures are related more geometrically, as well: F. Cohen shows an equivalence of categoried beteen the categories of Gerstenhaber algebras and of algebras over the homology of the little discs operad [5], and Getzler extends this to show an equivalence of categories between BV algebras and the homology of the framed little discs operad [15, Prop. 4.5]. There is a map of operads giving each unframed little disc the canonical framing, with the marked point at the top of the disc, and pullback in homology along this operad map then gives a BV algebra the canonical Gerstenhaber algebra structure above.

We also note that, since the equation governing a BV algebra is linear in Δ, it holds when Δ is replaced with $\lambda\Delta$ for $\lambda \in k$. Furthermore, the induced Gerstenhaber Lie bracket acquires the same scalar λ. Taking $\lambda = -1$, for example, if $(A, \cdot, \Delta)$ is a BV algebra, then so is $(A, \cdot, -\Delta)$.

A.2 Cofibrantly Generated Model Categories

Recall from Hovey [21, §§2.1, 2.3] the notions of a cofibrantly generated model category and the projective model structure on $\mathrm{Ch}(k)$, as well as the language and axioms of model categories. (In particular, note that Hovey requires a model category to have *functorial* factorizations of morphisms.) We use the following notation for classes of morphisms associated to cofibrantly generated model categories.

Notation A.2.1 Let I be a class of morphisms in a cocomplete category C. Let I-proj and I-inj denote the class of morphisms with the left and right lifting properties with respect to all morphisms in I, respectively, and denote a morphism in either class as an I-projective or an I-injective. Let I-cof (I-cofibrations) denote the class $(I\text{-inj})$-proj. Let I-cell (relative I-cell complexes) denote the class of maps formed by transfinite composition of pushouts along elements of I. ▪

Under appropriate set-theoretic conditions on the elements of I, the morphisms of C admits functorial factorizations via the small object argument, with the factors lying in I-cell and I-inj. See Hovey [21, §2.1] for more discussion of the set-theoretic issues involved.

Definition A.2.2 Suppose C is a model category. Then C is said to be *cofibrantly generated* if there are sets I and J of morphisms such that the domains of the morphisms of I and J are small relative to I-cell and J-cell and if the classes of fibrations and trivial fibrations are J-inj and I-inj. I and J are called the sets of *generating cofibrations* and *generating trivial cofibrations*. ▪

In a cofibrantly generated model category, then, I-cof and J-cof are the classes of cofibrations and trivial cofibrations, and each of their morphisms is a retract of a relative I-cell complex or relative J-cell complex. The category $\mathrm{Ch}(k)$ of chain complexes over k, in particular, admits a cofibrantly generated model category structure as follows [21, §2.2].

Definition A.2.3 For $n \in \mathbb{Z}$, let S^n denote the chain complex with k concentrated in degree n, as in Definition A.1.2, and let D^n denote the chain complex with $D^n_n = D^n_{n-1} = k$ and with differential $d : D^n_n \to D^n_{n-1}$ equal to id. Let $i_n : S^{n-1} \to D^n$ be the chain map taking S^{n-1}_{n-1} to D^n_{n-1} by the identity. Let $I = \{i_n\}_{n \in \mathbb{Z}}$ and let $J = \{0 \to D^n\}_{n \in \mathbb{Z}}$. ▪

Theorem A.2.4 $\mathrm{Ch}(k)$ is a cofibrantly generated model category with I as its set of generating cofibrations and J as its set of generating trivial cofibrations. The weak equivalences are the homology isomorphisms (i.e., quasi-isomorphisms), the fibrations are the surjections, and the cofibrations are those maps with the left lifting property with respect to all trivial fibrations. ∎

Furthermore, tensor product $\otimes$ and Hom-complexes give $\mathrm{Ch}(k)$ the structure of a symmetric monoidal model category [21, §4.2], so that $\otimes$ and Hom are suitably compatible with the model category structure.

A.3 A_∞ Algebras and Modules

A.3.1 A_∞ Algebras and Morphisms

We recall briefly from Keller [24] the fundamental notions of such algebras and their modules, although we treat chain complexes homologically instead of cohomologically and therefore must reverse the signs of some degrees.

Definition A.3.1 An A_∞-algebra over k is a graded k-module A_* with a sequence of graded k-linear maps $m_n : A^{\otimes n} \to A$ of degree $n-2$ for $n \geq 1$. These maps satisfy the quadratic relations

$$\sum_{\substack{r+s+t=n \\ s\geq 1}} (-1)^{r+st} m_{r+1+t}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0$$

for $n \geq 1$, where $r, t \geq 0$ and $s \geq 1$. ∎

The first of these relations, $m_1 m_1 = 0$, shows that m_1 is a differential, making A a chain complex. The second relation rearranges to

$$m_1 m_2 = m_2(m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1),$$

shows that $m_2 : A \otimes A \to A$ is a chain map with respect to the differential m_1. The third identity rearranges to

$$m_2(\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) = m_1 m_3 + m_3(m_1 \otimes \mathrm{id}^{\otimes 2} + \mathrm{id} \otimes m_1 \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes m_1),$$

which shows that m_2 is associative only up to chain homotopy, with m_3 the homotopy between the two different m_2 compositions. The higher relations then describe additional homotopy coherence data for the m_n maps. Such data also describe a degree-(-1) coderivation b of the DGC $B(k, A, k)$ with $b^2 = 0$; for more details on both of these perspectives, see [24, §3].

A differential graded algebra A determines an A_∞-algebra with $m_1 = d$, the differential of A, $m_2 = \mu$, and $m_n = 0$ for $n \geq 3$. Conversely, any A_∞-algebra with $m_n = 0$ for $n \geq 3$ is a DGA. All of the A_∞-algebras we consider will actually be DGAs. Likewise, there is a notion of a morphism of A_∞-algebras, but any morphism we consider between these DGAs will be an ordinary morphism of DGAs. In the coalgebra framework, a morphism of A_∞-algebras $A \to A'$ is equivalent to a morphism of DGCs $B(k, A, k) \to B(k, A', k)$.

A.3.2 A_∞ Modules and Morphisms

We turn to the definition of modules over A_∞ algebras and their morphisms.

Definition A.3.2 A (left) A_∞-module over an A_∞-algebra A is a graded k-module M with action maps $m_n^M : A^{\otimes(n-1)} \otimes M \to M$ of degree $n-2$ for $n \geq 1$, satisfying the same relation as in Definition A.3.1, with the m_j replaced with m_j^M where appropriate. ▪

This definition is equivalent to giving a degree-(-1) differential b_M with $b_M^2 = 0$ compatible with the left $B(k, A, k)$-comodule structure on $B(k, A, M)$. If A is a DGA and M is an ordinary A-module, then setting $m_1^M = d_M$, $m_2^M = a_M$, and $m_n^M = 0$ for $n \geq 3$ gives M the structure of an A_∞-module for A. All of the A_∞-modules we consider will arise this way.

We do need to consider morphisms of A_∞-module which do not arise from morphisms of ordinary modules, however.

Definition A.3.3 Let L, M be A_∞-modules for an A_∞-algebra A. A morphism $f : L \to M$ of A_∞-modules over A is a sequence of maps $f_n : A^{\otimes n-1} \otimes L \to M$ of degree $n-1$ satisfying the relations

$$\sum_{\substack{r+s+t=n \\ s \geq 1}} (-1)^{r+st} f_{r+t+1}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = \sum_{\substack{r+s=n \\ s \geq 1}} m_{r+1}(\mathrm{id}^{\otimes r} \otimes f_s),$$

for $n \geq 1$, where the m_i represent the multiplication maps for the A_∞-algebra A or the action maps for L and M. ▪

This definition is equivalent to specifying a morphism of DG comodules $B(L, A, k) \to B(M, A, k)$. While this perspective is convenient for more theoretical work, the explicit form of the maps and relations above is more suitable for checking that a proposed map is a morphism of modules.

When A is a DGA and L and M are A-modules, the m_i vanish for $i \geq 3$, and we obtain the simplified relations $d_M f_1 = f_1 d_L$ and

$$d_M f_n + (-1)^n f_n d_{A^{\otimes n-1} \otimes L} = -a_M(\mathrm{id} \otimes f_{n-1}) + \sum_{r=0}^{n-3} (-1)^r f_{n-1}(\mathrm{id}^{\otimes r} \otimes \mu \otimes \mathrm{id}^{\otimes n-r-2}) + (-1)^{n-2} f_{n-1}(\mathrm{id}^{\otimes n-2} \otimes a_L)$$

for $n \geq 2$. In this case, f_1 is a chain map $L \to M$ which commutes with the action of A only up to a prescribed homotopy, f_2. Each subsequent f_{n+1} gives a homotopy between different ways of interleaving f_n with the action of $n-1$ copies of A. We will use these concepts in Section 4.3 when comparing different adjoint module structures over $C_* G$.

Gugenheim and Munkholm [20] note that Tor exhibits functoriality with respect to such morphisms, and Keller [24] notes that this functoriality also $\mathrm{Ext}_A^*(-,-)$ also exhibits such extended functoriality. We state the form of the results we need below:

Proposition A.3.4 Let A be a DGA, and let L, M, N be A-modules. Suppose that A, L, and M are all cofibrant as chain complexes of k-modules. Then a morphism of A_∞-modules $f : L \to M$ induces maps

$$\mathrm{Tor}_*^A(N, f) : \mathrm{Tor}_*^A(N, L) \to \mathrm{Tor}_*^A(N, M) \quad \text{and} \quad \mathrm{Ext}_A^*(N, f) : \mathrm{Ext}_A^*(N, L) \to \mathrm{Ext}_A^*(N, M).$$

If the chain map $f_1 : L \to M$ is a quasi-isomorphism, these induced maps on Tor and Ext are also isomorphisms.

Proof: Since f is an A_∞-module morphism, it induces a morphism of $B(k, A, k)$-comodules $B(k, A, L) \to B(k, A, M)$. Since $B(A, A, L)$ can be described as the cotensor product

$$B(A, A, k) \,\square^{B(k,A,k)}\, B(k, A, L),$$

this morphism of comodules induces a chain map

$$B(A, A, L) \to B(A, A, M).$$

Since A, L, and M are cofibrant over k, these bar constructions provide cofibrant replacements for L and M as A-modules. Hence, applying the functors $N \otimes_A -$ and $\mathrm{Hom}_A(QN, -)$ and passing to homology induces the desired maps on Ext and Tor.

If f_1 is a quasi-isomorphism, then $B(A, A, f)$ is a weak equivalence between cofibrant A-modules and is therefore a homotopy equivalence. It therefore induces isomorphisms in Ext and Tor. ∎

Bibliography

[1] Hossein Abbaspour, Ralph L. Cohen, and Kate Gruher. String topology of Poincaré duality groups. *Geometry and Topology Monographs*, 13:1–10, 2008. Preprint at arXiv:math.AT/0511181.

[2] Ken Brown. *Cohomology of Groups*. Number 87 in Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1982.

[3] D. Burghelea and Z. Fiedorowicz. Cyclic homology and algebraic K-theory of spaces. II. *Topology*, 25(3):303–317, 1986.

[4] Moira Chas and Dennis Sullivan. String topology. Preprint, arXiv:math.GT/9911159, November 1999.

[5] Frederick R. Cohen. The homology of $\mathcal{C}_{n+1}$-spaces, $n \geq 0$. In *The Homology Of Iterated Loop Spaces*, number 533 in Lecture Notes in Mathematics, chapter III, pages 207–351. Springer, New York, 1976. Available at `http://www.math.uchicago.edu/~may/BOOKS/homo_iter.pdf`.

[6] Ralph L. Cohen, Kathryn Hess, and Alexander A. Voronov. *String Topology and Cyclic Homology*. Advanced Courses in Mathematics, CRM Barcelona. Birkhäuser, Berlin, 2006.

[7] Ralph L. Cohen and John D. S. Jones. A homotopy theoretic realization of string topology. *Math. Ann.*, 324(4):773–798, 2002.

[8] Ralph L. Cohen and John R. Klein. Umkehr maps. *Homology, Homotopy Appl.*, 11(1):17–33, 2009. Also available at arXiv:0711.0540 [math.AT].

[9] W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar. Duality in algebra and topology. *Adv. Math.*, 200(2):357–402, 2006.

[10] Yves Felix, Steve Halperin, and Jean-Claude Thomas. Differential graded algebras in topology. In I. M. James, editor, *Handbook of Algebraic Topology*, chapter 16, pages 829–865. Elsevier Science, Amsterdam, 1995.

[11] Yves Felix, Steve Halperin, and Jean-Claude Thomas. *Rational Homotopy Theory*. Number 205 in Graduate Texts in Mathematics. Springer, New York, 2001.

[12] Yves Félix, Luc Menichi, and Jean-Claude Thomas. Gerstenhaber duality in Hochschild cohomology. *J. Pure Appl. Algebra*, 199:43–59, 2005.

[13] Yves Félix and Jean-Claude Thomas. Rational BV-algebra in string topology. *Bull. Soc. Math. France*, 136(2):311–327, 2008. Preprint also available at arXiv:0705.4194 [math.AT].

[14] Murray Gerstenhaber. The cohomology structure of an associative ring. *Ann. of Math. (2)*, 78(2):267–288, 1963.

[15] E. Getzler. Batalin-Vilkovisky algebras and two-dimensional topological field theories. *Connemt. Math. Phys.*, 159(2):265–285, 1994.

[16] Victor Ginzburg. Calabi-Yau algebras. Preprint, arXiv:math.AG/0612139, December 2006.

[17] Paul G. Goerss and John R. Jardine. *Simplicial Homotopy Theory*. Number 174 in Progress in Mathematics. Birkhäuser Verlag, Basel, 1999.

[18] Thomas G. Goodwillie. Cyclic homology, derivations, and the free loopspace. *Topology*, 24(2):187–215, 1985.

[19] Kate Gruher. *String Topology of Classifying Spaces*. PhD thesis, Stanford University, 2007.

[20] V.K.A.M. Gugenheim and H.J. Munkholm. On the extended functoriality of Tor and Cotor. *J. Pure Appl. Algebra*, 4:9–29, 1974.

[21] Mark Hovey. *Model Categories*, volume 63 of *Mathematical surveys and monographs*. American Mathematical Society, Providence, RI, 1999.

[22] John D. S. Jones. Cyclic homology and equivariant homology. *Invent. Math.*, 87(2):403–423, 1987.

[23] Daniel M. Kan. On homotopy theory and c.s.s. groups. *Ann. of Math. (2)*, 68(1):38–53, 1958.

[24] Bernhard Keller. Introduction to A-infinity algebras and modules. *Homology Homotopy Appl.*, 3:1–35, 2001.

[25] John R. Klein. Poincaré immersions. *Forum Math.*, 11(6):717–734, 1999.

[26] John R. Klein. The dualizing spectrum of a topological group. *Math. Ann.*, 319(3):421–456, 2001.

[27] John R. Klein. Fiber products, Poincaré duality, and A_∞-ring spectra. *Proc. Amer. Math. Soc.*, 134(6):1825–1833, 2006.

[28] Jean-Louis Loday. *Cyclic Homology*, volume 301 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1992.

[29] Saunders Mac Lane. *Categories for the working mathematician*. Number 5 in Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1998.

[30] J. P. May. *The geometry of iterated loop spaces*. Number 271 in Lecture Notes in Mathematics. Springer, New York, 1972. Available in retyped form at `http://www.math.uchicago.edu/~may/BOOKS/gils.pdf`.

[31] John McCleary. *A User's Guide to Spectral Sequences*, volume 58 of *Cambridge studies in advanced mathematics*. Cambridge Univ. Press, Cambridge, 2001.

[32] James E. McClure and Jeffrey H. Smith. A solution of Deligne's Hochschild cohomology conjecture. In Donald M. Davis, Jack Morava, Goro Nishida, W. Stephen Wilson, and Nobuaki Yagita, editors, *Recent progress in homotopy theory*, number 293 in Contemporary Mathematics, pages 153–193. Amer. Math. Soc., Providence, RI, 2002.

[33] Luc Menichi. The cohomology ring of free loop spaces. *Homology Homotopy Appl.*, 3(1):193–224, 2001.

[34] Luc Menichi. Van den Bergh isomorphisms in string topology. Preprint, arXiv:0907.2105 [math.AT], July 2009.

[35] M. Rothenberg and N. E. Steenrod. The cohomology of classifying spaces of H-spaces. *Bull. Amer. Math. Soc. (N. S.)*, 71:872–875, 1965.

[36] Stefan Schwede and Brooke Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.

[37] Stefan Schwede and Brooke Shipley. Equivalences of monoidal model categories. *Algebr. Geom. Topol.*, 3:287–334, 2003.

[38] Brooke Shipley. $H\mathbb{Z}$-algebra spectra are differental graded algebras. *Amer. J. Math.*, 129(2):351–379, 2007.

[39] Michael Shulman. Homotopy limits and colimits and enirched homotopy theory. Preprint, arXiv:math.AT/0610194v2, 2006.

[40] Moss E. Sweedler. *Hopf algebras*. W. A. Benjamin, New York, 1969.

[41] Robert M. Switzer. *Algebraic Topology – Homology and Homotopy*. Classics in Mathematics. Springer-Verlag, Berlin, 2002.

[42] Hirotaka Tamanoi. A homotopy theoretic proof of the BV identity in loop homology. Preprint, arXiv:0706.1277 [math.AT], June 2007.

[43] Dmitri Tamarkin and Boris Tsygan. The ring of differential operators on forms in noncommutative calculus. In *Graphs and patterns in mathematics and theoretical physics*, volume 73 of *Proc. Sympos. Pure Math.*, pages 105–131, Providence, RI, 2005. Amer. Math. Soc.

[44] Thomas Tradler. The Batalin-Vilkovisky algebra on Hochschild cohomology induced by infinity inner products. *Ann. Inst. Fourier (Grenoble)*, 58(7):2351–2379, 2008.

[45] Thomas Tradler, Mahmoud Zeinalian, and Dennis Sullivan. Infinity structure of Poincaré duality spaces. *Algebr. Geom. Topol.*, 7:233–260, 2007.

[46] Dmitry Vaintrob. The string topology BV algebra, Hochschild cohomology and the Goldman bracket on surfaces. Preprint, arXiv:math.AT/0702859, February 2007.

[47] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge studies in advanced mathematics*. Cambridge Univ. Press, Cambridge, 1994.

[48] Craig Westerland. Equivariant operads, string topology, and Tate cohomology. Preprint, arXiv:math.AT/0605080, May 2006.