String Topology and the Based Loop Space

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String Topology

Fix $k$ a commutative ring. Let

- $M$ be a closed, $k$-oriented, smooth manifold of dimension $d$
- $LM = \text{Map}(S^1, M)$
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- a graded-commutative loop product $\circ$, from intersection product on $M$ and concatenation product on $\Omega M$
- a degree-1 operator $\Delta$ with $\Delta^2 = 0$, from the rotation of $S^1$
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Make $H_{*+d}(LM)$ a Batalin-Vilkovisky (BV) algebra:

- $\circ$ and $\Delta$ combine to produce a degree-1 Lie bracket on $H_{*+d}(LM)$, called the loop bracket.

Also an algebra over $H_*$ of the framed little discs operad. (Getzler)
Hochschild Homology and Cohomology

The Hochschild homology and cohomology of an algebra $A$ exhibit similar operations:

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Goal: relate these structures to string topology of $M$ for certain DG algebras associated to $M$:

- $C^*M$, cochains of $M$
- $C_*\Omega M$, chains on the based loop space $\Omega M$
Results

Theorem (M.)

Let $M$ be a connected, $k$-oriented Poincaré duality space of formal dimension $d$. Then Poincaré duality induces an isomorphism

$$D : HH^*(C_* \Omega M) \to HH_{*+d}(C_* \Omega M).$$
Results

Theorem (M.)

Let $M$ be a connected, $k$-oriented Poincaré duality space of formal dimension $d$. Then Poincaré duality induces an isomorphism

$$D : \text{HH}^*(C_* \Omega M) \to \text{HH}^{*+d}(C_* \Omega M).$$

Uses “derived” Poincaré duality (Klein, Dwyer-Greenlees-Iyengar):

- Generalize co/homology with local coefficients $E$ to allow $C_* \Omega M$-module coefficients
- Cap product with $[M]$ still induces an isomorphism

$$H^*(M; E) \to H^{*+d}(M; E).$$
Compatibility of Hochschild operations under $D$:

**Theorem (M.)**

$HH^*(C_* \Omega M)$ with the Hochschild cup product and the operator $-D^{-1}BD$ is a BV algebra, compatible with the Hochschild Lie bracket.
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**Theorem (M.)**

*When M is a manifold, the composite of D with the Goodwillie isomorphism $HH_\ast(C_\ast \Omega M) \to H_\ast(LM)$ takes this BV structure to that of string topology.*

Resolves an outstanding conjecture about string topology and Hochschild cohomology.
Previous Results

Pre-String Topology

- $HH_*(C_* \Omega X) \cong H_* LX$, taking $B$ to $\Delta$ (Goodwillie)
- $HH_*(C^* X) \cong H^* LX$, taking $B$ to $\Delta$, for $X$ 1-conn (Jones)
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Pre-String Topology

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String Topology and $C^* M$

- Thom spectrum $LM^{-TM}$ an algebra over the cactus operad (equivalent to the framed little discs operad) (Cohen-Jones)
- Cosimplicial model for $LM^{-TM}$ shows $HH^*(C^* M) \cong H_{*+d}(LM)$ as rings, $M$ 1-conn
- When char $k = 0$, $HH^*(C^* M)$ a BV algebra, isom to $H_{*+d}(LM)$, still need $M$ 1-conn (Félix-Thomas)
Previous Results

Koszul Duality

- $C$ a 1-conn finite-type coalgebra, $HH^*(C^\vee) \cong HH^*(\text{Cobar}(C))$, preserving the cup and bracket (Félix-Menichi-Thomas)
- When $\mathcal{M}$ 1-conn and $C = C_\ast \mathcal{M}$, gives $HH^*(C_\ast \mathcal{M}) \cong HH^*(C_\ast \Omega \mathcal{M})$
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- When $M$ 1-conn and $C = C_* M$, gives $HH^*(C^* M) \cong HH^*(C_* \Omega M)$

Group Rings

$G$ a discrete group, $M$ an aspherical $K(G,1)$ manifold.

- $H_{*+d}(G, kG^{\text{conj}})$ is a ring, isomorphic to $H_{*+d}(LM)$ (Abbaspour-Cohen-Gruher)
- $HH^*(kG)$ a BV algebra, isomorphic to $H_{*+d}(LM)$ (Vaintrob)

In this case, $\Omega M \simeq G$ so our result generalizes these ones
Homological Algebra of $C_\ast \Omega M$

Models for Homological Algebra

Replace $\Omega M$ with an equivalent top group so $C_\ast \Omega M$ a DGA

- $C_\ast \Omega M$ a cofibrant chain complex, so category of modules has cofibrantly generated model structure
- Two-sided bar constructions $B(\dashv, C_\ast \Omega M, \dashv)$ yield suitable models for Ext, Tor, and Hochschild co/homology of $C_\ast \Omega M$
Homological Algebra of $C_*\Omega M$

Models for Homological Algebra

Replace $\Omega M$ with an equivalent top group so $C_*\Omega M$ a DGA

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- Two-sided bar constructions $B(-, C_*\Omega M, -)$ yield suitable models for Ext, Tor, and Hochschild co/homology of $C_*\Omega M$

Rothenberg-Steenrod Constructions

Connect these bar constructions over $C_*\Omega M$ to topological settings

- $C_*M \simeq B(k, C_*\Omega M, k)$
- $C_*(F \times_G EG) \simeq B(C_*F, C_*G, k)$ for $G$ a top group
Derived Poincaré Duality

Co/homology with local coefficients: for $E$ a $k[\pi_1 M]$-module,

$$H_* (M; E) \cong \text{Tor}^{C_* \Omega M}_* (E, k), \quad H^* (M; E) \cong \text{Ext}^*_ {C_* \Omega M} (k, E)$$
Derived Poincaré Duality

Co/homology with local coefficients: for $E$ a $k[\pi_1 M]$-module,

$$H_\ast(M; E) \cong \text{Tor}_{C_* \Omega^M}(E, k), \quad H^\ast(M; E) \cong \text{Ext}^\ast_{C_* \Omega^M}(k, E)$$

- $E \otimes_{C_* \Omega^M}^L k$ and $R \text{Hom}_{C_* \Omega^M}(k, E)$ give “derived” co/homology with local coefficients in $E$ a $C_* \Omega^M$-module
Derived Poincaré Duality

Co/homology with local coefficients: for $E$ a $k[\pi_1 M]$-module,

$$H_*(M; E) \cong \text{Tor}^{C_* \Omega M}_{\ast}(E, k), \quad H^*(M; E) \cong \text{Ext}^{\ast}_{C_* \Omega M}(k, E)$$

- $E \otimes_{C_* \Omega M}^L k$ and $R \text{Hom}_{C_* \Omega M}(k, E)$ give “derived” co/homology with local coefficients in $E$ a $C_* \Omega M$-module
- View $[M] \in H_dM$ as a class in $\text{Tor}^{C_* \Omega M}_{\ast}(k, k)$. Then

$$\text{ev}[M] : R \text{Hom}_{C_* \Omega M}(k, E) \to E \otimes_{C_* \Omega M}^L k[d]$$

a weak equivalence for $E$ a $k[\pi_1 M]$-module
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- $E \otimes_{C_* \Omega M}^L k$ and $R\text{Hom}_{C_* \Omega M}(k, E)$ give “derived” co/homology with local coefficients in $E$ a $C_* \Omega M$-module
- View $[M] \in H_d M$ as a class in $\text{Tor}_d^{C_* \Omega M}(k, k)$. Then

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a weak equivalence for $E$ a $k[\pi_1 M]$-module
- Algebraic Postnikov tower, compactness of $k$ as a $C_* \Omega M$-module show a weak equivalence for all $C_* \Omega M$-modules $E$. 
Hochschild Homology and Cohomology

Let \( \text{Ad} \) be \( C_\ast \Omega M \) with \( C_\ast \Omega M \)-module structure from conjugation
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- Show Hochschild co/homology isomorphic to $\text{Ext}^\ast_{C_\ast \Omega M}(k, \text{Ad})$ and $\text{Tor}^\ast_{C_\ast \Omega M}(\text{Ad}, k)$
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Let $\text{Ad}$ be $C_* \Omega M$ with $C_* \Omega M$-module structure from conjugation

- Show Hochschild co/homology isomorphic to $\text{Ext}_{C_* \Omega M}^* (k, \text{Ad})$ and $\text{Tor}_{C_* \Omega M}^* (\text{Ad}, k)$
- Combine with derived Poincaré duality to get $D$:

$$
\begin{align*}
HH^* (C_* \Omega M) &\xrightarrow{\cong} \text{Ext}_{C_* \Omega M}^* (k, \text{Ad}) \\
\cong \downarrow D &\cong \downarrow \text{ev}[M] \\
HH_{*+d} (C_* \Omega M) &\xrightarrow{\cong} \text{Tor}_{C_* \Omega M}^* (\text{Ad}, k)
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\cong \quad \cong \\
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- Comes essentially from $B(G, G, G) \cong B(*, G, G \times G^{\text{op}})$ homeo plus Eilenberg-Zilber equivalences
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- Comes essentially from $B(G, G, G) \cong B(\ast, G, G \times G^{\text{op}})$ homeo plus Eilenberg-Zilber equivalences
- Need to insert SH-linear maps, though
Ring Structures

Show Hochschild cup product agrees with Chas-Sullivan loop product
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- Umkehr map from $\Delta_M$ makes $LM^{-TM}$ a ring spectrum, induces loop product in $H_*$ via Thom isom $LM^{-TM} \wedge Hk \simeq \Sigma^{-d} LM \wedge Hk$
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- Fiberwise Atiyah duality and simplicial techniques show that

$$LM^{-TM} \simeq \Gamma_M(S_M[LM]) \simeq S[\Omega M]^{h\Omega M} \simeq THH_S(S[\Omega M])$$
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- Similarly, $S[LM] \cong [\Omega M]_{h\Omega M} \cong THH^S(S[\Omega M])$
- Smash with $Hk$, pass to equivalent derived category $\text{Ho } k\text{-Mod}$ to recover chain-level equivalences
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- Use "cap-pairing" to show that $D$ isom given by Hochschild cap product against $z \in HH_d(C_* \Omega M)$:

$$D(f) = f \cap z$$
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- BV Lie bracket also agrees with Hochschild Lie bracket

$D$ and Goodwillie isom take $\cup$ to loop product and $-D^{-1}BD$ to $\Delta$
Future Directions

• Develop similar models for loop coproduct, string topology operations from fat graphs (Godin)
• Explore similar models using $C_\ast \Omega^n M$ for higher string topology on $H_\ast (\text{Map}(S^n, M))$, relate to Hu's work on $HH^\ast (n)(C^\ast M)$
• Connect this description of string topology to topological field theories via Cobordism Hypothesis

http://math.stanford.edu/~emalm/