Solutions to Final Practice Problems

1. Find derivatives of the following functions:

(a) $f(x) = 2x^3 + 4x^2 - 3x + 5$

Solution: $f'(x) = 6x^2 + 8x - 3$.

(b) $g(t) = \dfrac{t^4}{4} + \dfrac{1}{t^3} - \dfrac{2}{t^6}$

Solution: We rewrite $g(t)$ as $\dfrac{t^4}{4} + t^{-3} - 2t^{-6}$. Then $g'(t) = \frac{1}{4}(4t^3) - 3t^{-4} - 2(-6)t^{-7} = t^3 - 3t^{-4} + 12t^{-7}$. Rewriting with fractions,

$$g'(t) = t^3 - \frac{3}{t^4} + \frac{12}{t^7}.$$

(c) $r(u) = e^{u^2 \ln u}$

Solution: Using the chain rule and the product rule,

$$r'(u) = (u^2 \ln u)' e^{u^2 \ln u} = \left(2u \ln u + u^2 \frac{1}{u}\right) e^{u^2 \ln u} = (2u \ln u + u) e^{u^2 \ln u}.$$

(d) $Q(z) = \dfrac{e^z}{2 + \sqrt{z}}$

Solution: Using the quotient rule,

$$Q'(z) = \frac{e^z(2 + \sqrt{z}) - e^z \frac{1}{2\sqrt{z}}}{(2 + \sqrt{z})^2} = \frac{e^z(4\sqrt{z} + 2z - 1)}{2\sqrt{z}(2 + \sqrt{z})^2}.$$

(e) $m(y) = 2^{3y - y^2}$

Solution: Using the chain rule and the derivative rule for a^x,

$$m'(y) = (\ln 2)(3 - 2y) 2^{3y - y^2}.$$

2. Compute the values of the following definite integrals:

(a) $\int_{-1}^{4} 2x + 1\, dx$

Solution: An antiderivative of $2x + 1$ is $x^2 + x$, so

$$\int_{-1}^{4} 2x + 1\, dx = x^2 + x\Big|_{-1}^{4} = (16 + 4) - (1 - 1) = 20.$$

(b) $\int_{1}^{3} x^3 - 6x^2 + 12x - 8\, dx$

Solution: An antiderivative of $x^3 - 6x^2 + 12x - 8$ is $\frac{1}{4}x^4 - 2x^3 + 6x^2 - 8x$, so

$$\begin{aligned}\int_{1}^{3} x^3 - 6x^2 + 12x - 8\, dx &= \frac{1}{4}x^4 - 2x^3 + 6x^2 - 8x\Big|_{1}^{3} \\ &= \left(\frac{81}{4} - 54 + 54 - 24\right) - \left(\frac{1}{4} - 2 + 6 - 8\right) = 0.\end{aligned}$$

(c) $\int_{0}^{10} e^{0.2t}\, dt$

Solution: An antiderivative for $e^{0.2t}$ is $\dfrac{1}{0.2}e^{0.2t} = 5e^{0.2t}$. Then

$$\int_{0}^{10} e^{0.2t}\, dt = 5e^{0.2t}\Big|_{0}^{10} = 5e^2 - 5 \approx 31.945.$$

3. Find the general antiderivatives of the following functions:

(a) $f(x) = 6x^2 - 4x + 3$

Solution: Using the power rule, the general antiderivative of $6x^2 - 4x + 3$ is $2x^3 - 2x^2 + 3x + C$.

(b) $g(t) = e^{t/6} - \dfrac{1}{t^3}$

Solution: Writing $g(t) = e^{t/6} - t^{-3}$, the general antiderivative of $g(t)$ is

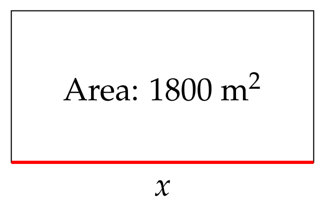
$$\frac{1}{1/6}e^{t/6} - \frac{1}{-2}t^{-2} + C = 6e^{t/6} - \frac{1}{2t^2} + C.$$

(c) $h(z) = \frac{3}{z} + \frac{1}{\sqrt[3]{z}}$

Solution: Write $h(z) = 3\frac{1}{z} + z^{-1/3}$. Then the general antiderivative of $h(z)$ is

$$3 \ln z + \frac{1}{2/3} z^{2/3} + C 3 \ln z + \frac{3}{2} z^{2/3} + C.$$

4. We are building a rectangular garden of area 1800 m^2. On three of the sides, we will use fence that costs \$20 per meter, and on the last side we will use fence that costs \$60 per meter. Let x be the length of the side of the garden with the more expensive fence.



(a) Find an expression for the cost C of the fence in terms of x.

Solution: Let y be the length of the other side of the garden. Then the area of the garden is xy, so $xy = 1800$, and $y = \frac{1800}{x}$. The cost of one x side is \$60 per meter and the cost of the other x side and the two y sides is \$20, so the total cost of the fence in terms of x is

$$C(x) = (60 + 20)x + 2(20)y = 80x + 40\frac{1800}{y} = 80x + \frac{72{,}000}{x}.$$

(b) Find the value of x that minimizes the cost of the fence.

Solution: We find the global minimum value of $C(x)$. First, we find $C'(x)$:

$$C'(x) = 80 - \frac{72{,}000}{x^2}.$$

Setting $C'(x) = 0$, $80 - \frac{72{,}000}{x^2} = 0$, so $80x^2 = 72{,}000$. Dividing by 80, $x^2 = 900$, so $x = 30$ (we take only the positive solution). We note that for $x < 30$, $C'(x)$ is negative, and for $x > 30$, it is positive, so $x = 30$ actually corresponds to a global minimum.

(c) At the minimum cost, what are the dimensions of the garden, and how much does the entire fence cost?

Solution: When $x = 30$, $y = \frac{1800}{30} = 60$. Therefore, the cost is

$$C(30) = 80(30) + 40(60) = 4800.$$

5. Let $f(x) = \frac{1}{4}x^4 - x^3 - 2x^2 + 3$.

(a) Find $f'(x)$ and $f''(x)$.

Solution: Taking the derivative, $f'(x) = x^3 - 3x^2 - 4x$ and $f''(x) = 3x^2 - 6x - 4$.

(b) Find the critical points of $f(x)$.

Solution: We solve $f'(x) = 0$, so $x^3 - 3x^2 - 4x = 0$. Factoring, $x(x+1)(x-4) = 0$, so $x = 0$, $x = -1$, and $x = 4$ are the critical points of $f(x)$.

(c) Characterize each critical point as a local minimum, local maximum, or neither. Justify your answers.

Solution: We first try the second derivative test, so we evaluate $f''(x)$ at each critical point:

$$f''(-1) = 5, \qquad f''(0) = -4, \qquad f''(4) = 20.$$

Therefore, $f(x)$ has a local maximum at $x = 0$ and local minima at $x = -1$ and $x = 4$.

(d) Find the intervals on which $f(x)$ is increasing and on which $f(x)$ is decreasing.

Solution: We use the characterization of the critical points above.

- Since the leftmost critical point, $x = -1$, is a local minimum, $f(x)$ is decreasing for $x < -1$ and increasing on $-1 < x < 0$.
- Since the next critical point is a local maximum, $f(x)$ is decreasing on $0 < x < 4$.
- Since the last critical point is a local minimum, $f(x)$ is increasing for $x > 4$.

Hence, the intervals of increase are $(-1, 0)$ and $(4, \infty)$, and the intervals of decrease are $(-\infty, -1)$ and $(0, 4)$.

(e) Find the *global* maximum and minimum values of $f(x)$ on the interval $[-2, 4]$, and the x-values where they occur.

Solution: We check the value of $f(x)$ itself at the critical points and at the endpoint $x = -2$ (the other endpoint, $x = 4$, is already listed as a critical point):

$$f(-2) = \frac{16}{4} - (-8) - 8 + 3 = 7, \qquad f(-1) = \frac{1}{4} - (-1) - 2 + 3 = \frac{9}{4},$$
$$f(0) = 0 + 0 + 0 + 3 = 3, \qquad f(4) = \frac{4^4}{4} - 64 - 32 + 3 = -29.$$

The lowest value, -29, occurs at $x = 4$, and so is the global minimum value. The highest value, 7, occurs at $x = -2$, and so is the global maximum value.

(f) Find the inflection points of $f(x)$. Justify your answers.

Solution: To find the inflection points, we solve $f''(x) = 0$. Then $3x^2 - 6x - 4 = 0$, so we use the quadratic formula:

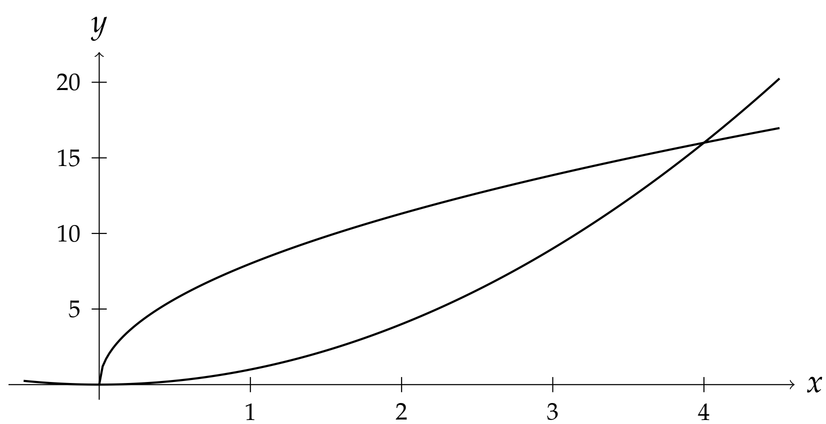
$$x = \frac{6 \pm \sqrt{36+48}}{6} = 1 \pm \sqrt{\frac{7}{3}}.$$

The points $1 - \sqrt{\frac{7}{3}}$ and $1 + \sqrt{\frac{7}{3}}$ lie between the critical points $-1, 0$, and 4. Since the sign of the second derivative switches at each critical point, the concavity does change at each of these x values, so both x values above give inflection points.

6. Find the area of the region enclosed by the curves $y = x^2$ and $y = 8\sqrt{x}$.

Solution: We find where $y = x^2$ and $y = 8\sqrt{x}$ intersect. To do so, we set the y-values equal to each other, which gives $x^2 = 8\sqrt{x}$. Squaring both sides, $x^4 = 64x$. This has $x = 0$ as a solution. If we instead assume $x \neq 0$, dividing by x gives $x^3 = 64$, so $x = \sqrt[3]{64} = 4$. Hence, the two curves intersect at $x = 0$ and $x = 4$.

We note that $8\sqrt{x}$ is greater than x^2 between 0 and 4: for example, at $x = 1$, $8\sqrt{x} = 8$, but $x^2 = 1$. We can also see that from a sketch of the two curves:



Therefore, $8\sqrt{x}$ is the upper curve of the region, and x^2 is the lower curve. The area is then given by

$$\int_0^4 8\sqrt{x} - x^2\,dx = 8\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\Big|_0^4 = 8\frac{2}{3}(8) - \frac{64}{3} - (0-0) = \frac{64}{3}.$$

7. Let $f(x) = x^3 - \dfrac{1}{x^2}$.

(a) Find the general antiderivative $G(x)$ of $f(x)$.

Solution: Since $f(x) = x^3 - x^{-2}$, the general antiderivative of $f(x)$ is

$$G(x) = \frac{1}{4}x^4 - \frac{1}{-1}x^{-1} + C = \frac{1}{4}x^4 + \frac{1}{x} + C.$$

(b) Find the antiderivative $G(x)$ of $f(x)$ satisfying $G(2) = \dfrac{5}{2}$.

Solution: We find the value of C that makes $G(2) = \frac{5}{2}$:

$$\frac{5}{2} = G(2) = \frac{2^4}{4} + \frac{1}{2} + C = 4 + \frac{1}{2} + C.$$

Then $C = \frac{5}{2} - 4 - \frac{1}{2} = -2$, so the specific antiderivative is $\frac{1}{4}x^4 + \frac{1}{x} - 2$.

8. Let $g(x) = x^{4/3}$.

(a) Find $g'(x)$.

Solution: Using the power rule, $g'(x) = \dfrac{4}{3}x^{1/3}$.

(b) Find the equation of the tangent line to the graph $y = x^{4/3}$ at $x = 8$.

Solution: The tangent line goes through the point $(8, g(8)) = (8, 16)$ on the graph. Its slope is $g'(8) = \frac{4}{3}8^{1/3} = \frac{8}{3}$. From the point-slope formula, the line is given by

$$y = 16 + \frac{8}{3}(x - 8).$$

(c) Use the tangent line to approximate $(8.03)^{4/3}$.

Solution: We plug $x = 8.03$ into the equation for the tangent line to get

$$y = 16 + \frac{8}{3}(8.03 - 8) = 16 + 0.08 = 16.08.$$

Therefore, we estimate that $(8.03)^{4/3} \approx 16.08$.

9. A water pipe bursts. The flow through the burst pipe wall is given, in liters per second, by $F(t)$, where t is the time in seconds since the pipe burst. A table of these flow rates is given below:

t (s)	0	1	2	3	4
$F(t)$ (l/s)	40	30	24	18	14

Using the average of a left-hand and a right-hand Riemann sum, estimate the volume of water that escapes the pipe during the first 4 seconds.

Solution: For these Riemann sums, the time interval between data points is $\Delta t = 1$. Hence, the left-hand sum is

$$\sum_{i=0}^{3} F(i) \cdot 1 = 40 + 30 + 24 + 18 = 112,$$

and the right-hand sum is

$$\sum_{i=1}^{4} F(i) \cdot 1 = 30 + 24 + 18 + 14 = 86.$$

We typically get the best estimate from taking the average of the left- and right-hand sums, which in this case is $\dfrac{112 + 86}{2} = 99$. Therefore, we estimate that 99 liters of water escapes in the first 4 seconds.

10. We sell bushels of apples from our farm in Riverhead, NY. We discover that if we sell a bushel at \$20, we can sell 2000 bushels, but if we decrease the price to \$18, we sell 2400.

(a) Find the price $p(q)$ as a function of the quantity sold, q.

Solution: We note that when the price decreases by 2, the quantity increases by 400. Therefore,

$$\frac{p - 20}{q - 2000} = \frac{\Delta p}{\Delta q} = \frac{-2}{400} = -\frac{1}{200}.$$

We isolate p by multiplying by $q - 2000$ on both sides and adding 20:

$$p(q) = 20 - \frac{1}{200}(q - 2000) = 20 - \frac{1}{200}q + 10 = 30 - \frac{1}{200}q.$$

(b) Find the revenue $R(q)$ as a function of q.

Solution: The revenue is given by the price times the quantity, so it is

$$R(q) = qp(q) = q\left(30 - \frac{1}{200}q\right) = 30q - \frac{1}{200}q^2.$$

(c) Find the quantity q that maximizes the revenue. What is the price we should charge to sell that quantity?

Solution: To maximize the revenue, we compute $R'(q)$, set it to 0, and solve for q:

$$R'(q) = 30 - \frac{1}{200}(2q) = 30 - \frac{1}{100}q$$

Then $30 - \frac{1}{100}q = 0$, so $q = 30(100) = 3000$. Since $R''(q) = -\frac{1}{100}$, this corresponds to a local maximum, and is in fact a global maximum since $R(q)$ describes a concave-down parabola. The price at this quantity is $p(3000) = 30 - \frac{1}{200}(3000) = 15$ dollars.

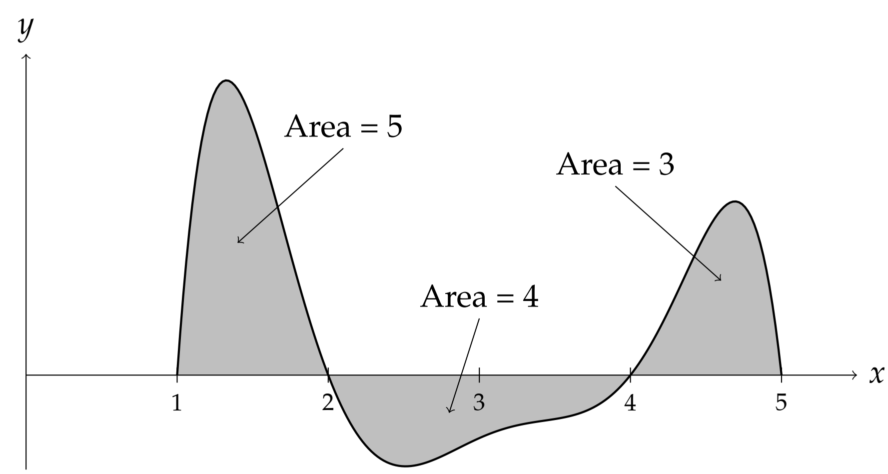
(d) We calculate our costs to be $C(q) = 5000 + 10q$. Find the quantity that maximizes the profit that we make. What is the price we should charge, and what is our profit?

Solution: To maximize profits, we now solve $R'(q) = C'(q)$ for q. Since $C'(q) = 10$,

$$30 - \frac{1}{100}q = 10,$$

so $q = 100(30 - 10) = 2000$. The price to sell this quantity is $p(2000) = 30 - \frac{1}{200}(2000) = 20$ dollars per bushel. The revenue is then $(2000)(20) = 40{,}000$ dollars, and the cost at this quantity is $C(2000) = 25{,}000$, so the profit is $40{,}000 - 25{,}000 = 15{,}000$ dollars.

11. Below is the graph of a function $g(x)$ on the interval $[1, 5]$.



(a) Find $\displaystyle\int_1^4 g(x)\,dx$.

Solution: The value of this integral is the signed area from $x = 1$ to $x = 4$. The area above the x-axis from 1 to 2 contributes 5, while the area below the x-axis from 2 to 4 contributes -4, for a total of $5 + (-4) = 1$.

(b) Find $\int_2^5 g(x)\,dx$.

Solution: The value of this integral is the signed area from $x = 2$ to $x = 5$. The area below the x-axis from 2 to 4 contributes -4, while the area above the x-axis from 4 to 5 contributes 3, for a total of $(-4) + 3 = -1$.

(c) Find $\int_1^5 g(x)\,dx$.

Solution: The value of this integral is the signed area from $x = 1$ to $x = 5$, so is $5 + (-4) + 3 = 4$.

(d) Find the total area of the shaded region enclosed by the curve and the x-axis.

Solution: The total shaded area is the (unsigned) area of the three regions, which is $5 + 4 + 3 = 12$.

(e) Find the average value of $g(x)$ from $x = 1$ to $x = 5$.

Solution: The average value of $g(x)$ on $[1, 5]$ is

$$\frac{1}{5-1}\int_1^5 g(x)\,dx = \frac{1}{4}(4) = 1,$$

using the computation from part (c).

12. Let $z(t) = \dfrac{36}{(2z+1)^2}$.

(a) Find the left-hand Riemann sum of $z(t)$ from 0 to $\frac{3}{2}$ with $n = 3$.

Solution: For this left-hand Riemann sum, $a = 0$, $b = \frac{3}{2}$, and $n = 3$, so $\Delta t = \frac{1}{2}$. Then $t_i = 0 + i\frac{1}{2} = \frac{i}{2}$, so the sum is

$$\sum_{i=0}^{2} z(t_i)\Delta t = \frac{1}{2}\left(z(0) + z\left(\frac{1}{2}\right) + z(1)\right) = \frac{1}{2}\left(\frac{36}{1} + \frac{36}{4} + \frac{36}{9}\right) = \frac{49}{2}.$$

(b) Find the right-hand Riemann sum of $z(t)$ from 0 to $\frac{3}{2}$ with $n = 3$.

Solution: For this right-hand Riemann sum, $a = 0$, $b = \frac{3}{2}$, and $n = 3$, so $\Delta t = \frac{1}{2}$. Then $t_i = 0 + i\frac{1}{2} = \frac{i}{2}$, so the sum is

$$\sum_{i=1}^{3} z(t_i)\Delta t = \frac{1}{2}\left(z\left(\frac{1}{2}\right) + z(1) + z\left(\frac{3}{2}\right)\right) = \frac{1}{2}\left(\frac{36}{4} + \frac{36}{9} + \frac{36}{16}\right) = \frac{61}{8}.$$

(c) Use these Riemann sums to estimate $\int_0^{3/2} z(t)\,dt$.

Solution: We average the values of these Riemann sums:

$$\frac{1}{2}\left(\frac{49}{2}+\frac{61}{8}\right)=\frac{1}{2}\left(\frac{257}{8}\right)=\frac{257}{16}=16.0625.$$

(d) An antiderivative of $z(t)$ is $w(t)=-\dfrac{18}{2z+1}$. Compute the exact value of $\int_0^{3/2} z(t)\,dt$. What is the error in your estimate from part (c)?

Solution: Give this antiderivative,

$$\int_0^{3/2} z(t)\,dt=-\frac{18}{2z+1}\bigg|_0^{3/2}=-\frac{18}{4}+18=\frac{27}{2}=13.5.$$

The error is $16.0625-13.5=2.5625$.