# Midterm #1 — September 26, 2011, 8:30 to 10:00 PM

Name:       <u>       Solution Key       </u>

Circle your recitation:

R02 (Chira, Mon)      R03 (Chira, Wed)      R04 (Marcelo, Tue)

- **You have a maximum of $1\frac{1}{2}$ hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids are allowed.

- Read each question carefully. Show your work and justify your answers for full credit. You do not need to simplify your answers unless instructed to do so.

- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

**Grading**

| | |
|---|---|
| **1** | /15 |
| **2** | /18 |
| **3** | /16 |
| **4** | /12 |
| **5** | /15 |
| **6** | /14 |
| **7** | /10 |
| **Total** | /100 |

**1.** (*15 points*) Below is the graph of a linear function $L(x)$.
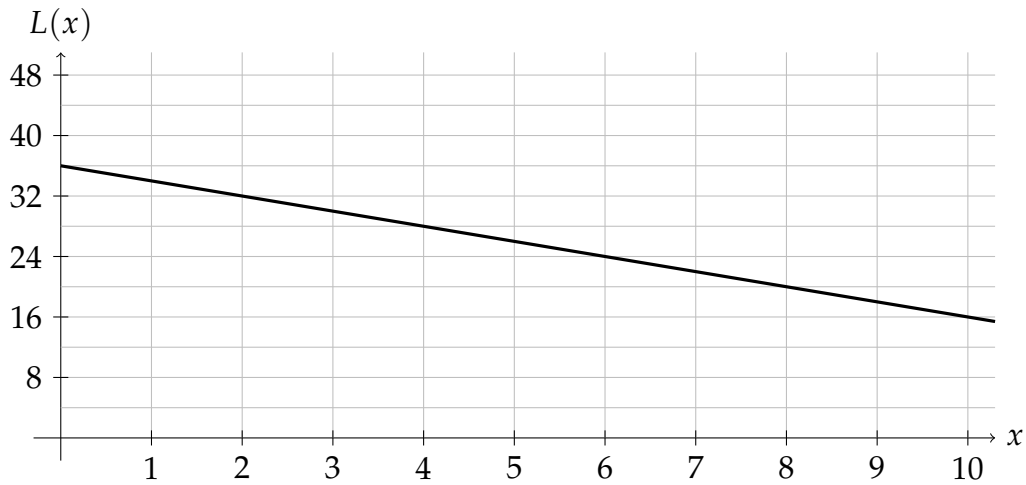


(a) Estimate $L(2)$ and $L(6)$.

*Solution*: (*5 points*) From the graph, $L(2) = 32$ and $L(6) = 24$.

(b) What is the slope of the line?

*Solution*: (*5 points*) Using the values from part (a), we calculate the slope to be

$$m = \frac{24 - 32}{6 - 2} = \frac{-8}{4} = -2.$$

(c) Write a formula for $L(x)$.

*Solution*: (*5 points*) Using the slope from part (b) and the first data point, $L(x)$ is

$$L(x) = -2(x - 2) + 32 = -2x + 4 + 32 = 36 - 2x.$$

Alternately, we can find the $y$-intercept by inspection of the graph.

**2.** (*18 points*)   After $t$ years, a savings account has a dollar value given by

$$V(t) = 10{,}000(1.1)^t.$$

(a) What is the initial value of the investment?

*Solution*: (*4 points*) The initial value is $10,000.

(b) What is the growth rate, as a percentage?

*Solution*: (*4 points*) The growth factor is 1.1, so subtracting 1 and converting to a percentage gives the growth rate:

$$r = 1.1 - 1 = 0.1 = 10\%.$$

(c) What is the value of the investment after 2 years?

*Solution*: (*5 points*) After 2 years, the investment is worth

$$V(2) = 10{,}000(1.1)^2 = 12{,}100 \text{ dollars.}$$

(d) At what time $t$ would the investment reach $30,000?   (You do not need to give a numerical answer, but it should be something you could enter into a calculator to get one.)
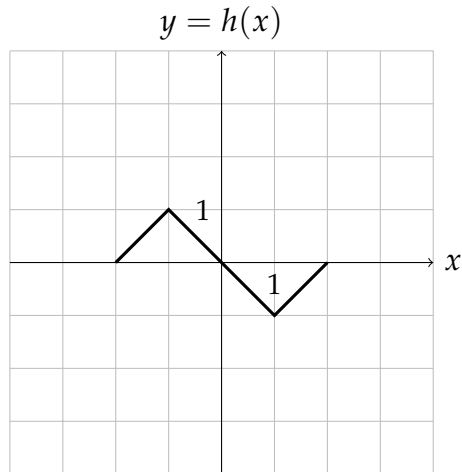
*Solution*: (*5 points*) We set $V(t) = 30{,}000$ and solve for $t$:
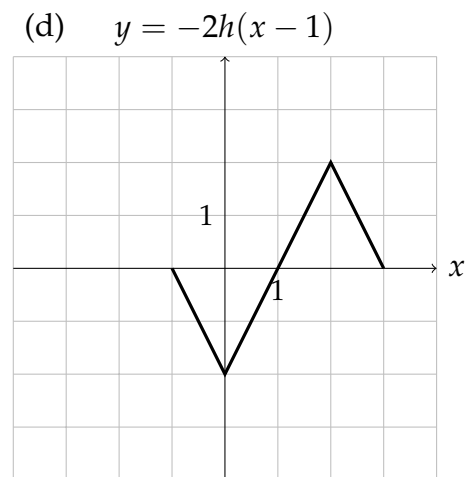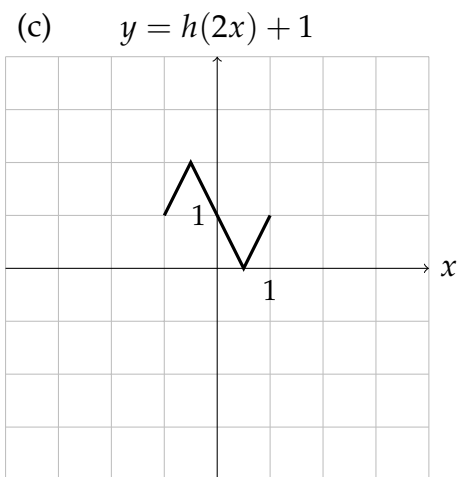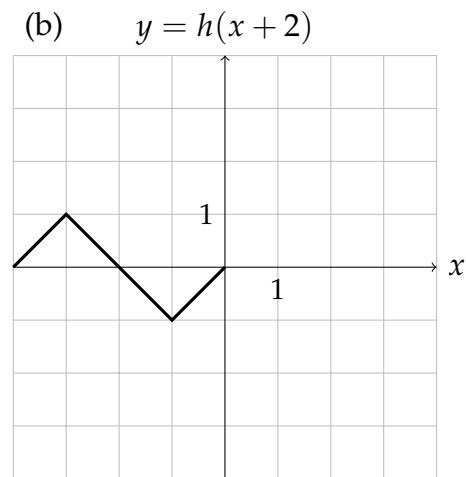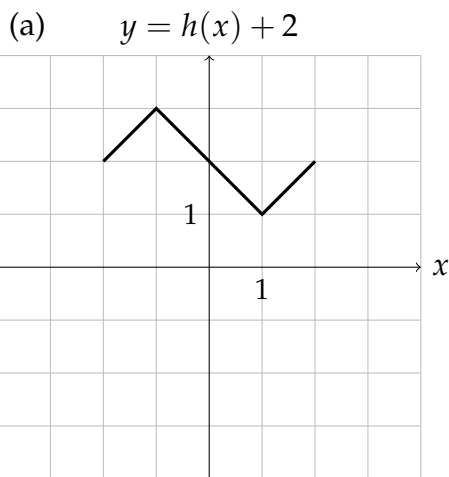
$$10{,}000(1.1)^2 = 30{,}000$$
$$(1.1)^2 = \frac{30{,}000}{10{,}000} = 3$$
$$t = \log_{1.1}(3) = \frac{\ln 3}{\ln 1.1}$$

**3.** (*16 points*)   Below is the graph of a function $h(x)$.

$$y = h(x)$$

Sketch graphs of the following functions on the axes provided:

(a)      $y = h(x) + 2$

(b)      $y = h(x + 2)$

(c)      $y = h(2x) + 1$

(d)      $y = -2h(x - 1)$

**4.** (*12 points*)   After the 2010 census, the city of Greenpoint has a population of 27,000, and the nearby city of Darbyville has a population of 82,000.

(a) Suppose Greenpoint grows at an *annual* rate of 5%. Write a formula for its population $t$ years after 2010.

*Solution*: (*6 points*) Since Greenpoint grows at 5% per year, its growth factor $a$ is $1 + 0.05 = 1.05$. Since it starts at an initial population of $P_0 = 27{,}000$, the population at time $t$ is

$$P(t) = P_0 a^t = 27{,}000(1.05)^t.$$

(b) Darbyville instead loses population at a *continuous* rate of 7% per year. Write a formula for its population $t$ years after 2010.

*Solution*: (*6 points*) Darbyville's continuous growth rate is $k = -0.07$, and its initial population is $P_0 = 82{,}000$. Therefore, its population at time $t$ is

$$P(t) = P_0 e^{kt} = 82{,}000 e^{-0.07t}.$$

**5.** (*15 points*)  Define the functions $f(x) = x^2 + 3$ and $g(x) = 2x - 3$. Find formulas for

(a) $f(g(x))$

*Solution*: (*5 points*) $f(g(x)) = f(2x - 3) = (2x - 3)^2 + 3$. Simplifying,

$$f(g(x)) = 4x^2 - 12x + 12.$$

(b) $g(f(x))$

*Solution*: (*5 points*) $g(f(x)) = g(x^2 + 3) = 2(x^2 + 3) - 3$. Simplifying,

$$g(f(x)) = 2x^2 + 3.$$

(c) $f(f(x))$

*Solution*: (*5 points*) $f(f(x)) = f(x^2 + 3) = (x^2 + 3)^2 + 3$. Simplifying,

$$f(f(x)) = x^4 + 6x^2 + 12.$$

**6.** (*14 points*) We are tracking the location of a railroad car traveling along a track from Santa Fe to Houston by measuring its distance along the track from Santa Fe.

| Time | 1 pm | 2 pm | 3 pm | 4 pm | 5 pm |
|---|---|---|---|---|---|
| Position (miles) | 440 | 504 | 567 | 628 | 687 |

(a) What is the average velocity of the car between 1 pm and 2 pm? Include units.

*Solution*: (*5 points*) Between 1 pm and 2 pm, the average velocity is

$$v = \frac{\Delta p}{\Delta t} = \frac{504 - 440}{2 - 1} = \frac{64}{1} = 64.$$

The distance is in miles and the time is in hours, so the velocity is in miles per hour (mph).

(b) What is the average velocity of the car between 3 pm and 5 pm? Include units.

*Solution*: (*5 points*) Between 3 pm and 5 pm, the average velocity is

$$v = \frac{\Delta p}{\Delta t} = \frac{687 - 567}{5 - 3} = \frac{120}{2} = 60,$$

again in miles per hour.

(c) Could the position of the car be a linear function of time? Explain your answer.

*Solution*: (*4 points*) No: if the position were a linear function, the average velocity would be constant, but instead it changes from 64 to 60.

**7.** (*10 points*)   We are buying steel for a new building. Acme Steel offers steel at $800 per ton and will charge us an extra $1000 to deliver our order. Bethlehem Steel sells steel for only $700 a ton but charges an extra $2000 to deliver it, as they are located farther away.

(a) Write functions $A(x)$ and $B(x)$ that give the cost of buying $x$ tons of steel from Acme and from Bethlehem, respectively.

*Solution*: (*5 points*) From Acme, the cost of $x$ tons of steel is $800x$, and the delivery cost is a flat 1000, so the total cost is

$$A(x) = 800x + 1000.$$

From Bethlehem, the cost of $x$ tons of steel is $700x$, and the delivery cost is a flat 2000, so the total cost is

$$B(x) = 700x + 2000.$$

(b) Under what circumstances should you buy steel from Acme? From Bethlehem? Explain your decision.

*Solution*: (*5 points*) We compute the price difference $A(x) - B(x)$ between Acme and Bethlehem for $x$ tons of steel:

$$A(x) - B(x) = (800x + 1000) - (700x + 2000) = 100x - 1000.$$

When Acme is less expensive, $A(x) < B(x)$, so this quantity is negative. This happens when

$$100x - 1000 < 0 \quad \Rightarrow \quad 100x < 1000 \quad \Rightarrow \quad x < \frac{1000}{100} = 10$$

Hence, if we need less than 10 tons, we should purchase steel from Acme. On the other hand, if we need more than 10 tons of steel, then this quantity is positive, so Bethlehem will be less expensive. If we need exactly 10 tons, both companies cost the same, so we can buy from either.