Homework #3 Solutions

Problems

• Section 1.6: 10, 20, 24, 28, 36
• Section 1.7: 2, 6, 16, 20
• Section 1.8: 2, 8, 14, 20, 22, 28

1.6.10. Solve for t using natural logarithms if \( 10 = 6e^{0.5t} \).

Solution: We isolate t:

\[
\frac{10}{6} = e^{0.5t}
\]

\[
\ln\left(\frac{10}{6}\right) = \ln(e^{0.5t}) = 0.5t
\]

\[
t = \frac{\ln\left(\frac{10}{6}\right)}{0.5} = 2 \ln\left(\frac{5}{3}\right)
\]

Evaluating this numerically, \( t \approx 1.022 \).

1.6.20. The function \( P = 3.2e^{0.03t} \) represent exponential growth or decay. What is the initial quantity? What is the growth rate? State if the growth rate is continuous.

Solution: The initial quantity is 3.2, and the (continuous) growth rate is 0.03.

1.6.24. Write the function \( P = 2e^{-0.5t} \) in the form \( P = P_0a^t \). Does the function represent exponential growth or decay?

Solution: The continuous growth rate \( k \) is \(-0.5\), so the growth factor \( a \) is \( e^{-0.5} \approx 0.607 \). Since \( k < 0 \), this function represents decay.

1.6.28. Put the function \( P = 10(1.7)^t \) in the form \( P = P_0e^{kt} \).

Solution: In this case, the growth factor \( a \) is 1.7, so \( k = \ln(1.7) \approx 0.531 \).
1.6.36. The gross world product is \( W = 32.4(1.036)^t \), where \( W \) is in trillions of dollars and \( t \) is years since 2001. Find a formula for gross world product using a continuous growth rate.

Solution: The growth factor is \( a = 1.036 \), so the continuous growth rate is \( k = \ln(1.036) \approx 0.0354 \). Hence, the new formula is

\[
W = 32.4e^{0.0354t}.
\]

1.7.2. The half-life of nicotine in the blood is 2 hours. A person absorbs 0.4 mg of nicotine by smoking a cigarette. Fill in the following table with the amount of nicotine in the blood after \( t \) hours. Estimate the length of time until the amount of nicotine is reduced to 0.04 mg.

Solution:

<table>
<thead>
<tr>
<th>( t ) (hours)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nicotine (mg)</td>
<td>0.4</td>
<td>0.2</td>
<td>0.1</td>
<td>0.05</td>
<td>0.025</td>
<td>0.0125</td>
</tr>
</tbody>
</table>

A formula describing the amount of nicotine at time \( t \) is \( N(t) = 0.4(2)^{-t/2} \). Then 0.04 = 0.4(2)^{-t/2}, so 0.1 = 2^{-t/2}, and \(-t/2 = \log_2(0.1)\). Finally, \( t = -2\log_2(0.1) \approx 6.64 \) hours.

1.7.6. Suppose 1000 is invested in an account paying interest at a rate of 5.5% per year. How much is in the account after 8 years if the interest is compounded

(a) Annually?

(b) Continuously?

Solution (a): The growth rate is 0.055, so the amount is \( 1000(1.055)^8 \approx 1534.69 \).

Solution (b): The continuous growth rate is 0.055, so the amount is \( 1000e^{0.055(8)} \approx 1552.71 \).

1.7.16. The antidepressant fluoxetine (or Prozac) has a half-life of about 3 days. What percentage of a dose remains in the body after one day? After one week?

Solution: We write a function \( P(t) \) that gives the proportion of Prozac remaining after time \( t \). Since the half-life is 3 days, \( P(t) = 2^{-t/3} \). Then \( P(1) = 2^{-1/3} \approx 0.794 = 79.4\% \) is the percentage remaining after 1 day, and \( P(7) = 2^{-7/3} \approx 0.198 = 19.8\% \) is the percentage remaining after 7 days.
1.7.20. The number of people living with HIV infections increased worldwide approximately exponentially from 2.5 million in 1985 to 37.8 million in 2003.

(a) Give a formula for the number of HIV infections, \( H \), (in millions) as a function of years, \( t \), since 1985. Use the form \( H = H_0 e^{kt} \). Graph the function.

(b) What was the yearly continuous percent change in the number of HIV infections between 1985 and 2003?

Solution (a): We have that \( H_0 = 2.5 \). In 2003, \( t = 2003 - 1985 = 18 \), so we have \( 37.8 = 2.5e^{18k} \). Then \( k = \ln \left( \frac{37.8}{2.5} \right) / 18 \approx 0.151 \), so the formula is

\[ H(t) = 2.5e^{0.151t}. \]

The graph of this function is as follows:

Solution (b): The continuous percentage change is 15.1% per year.

1.8.2. If \( f(x) = x^2 + 1 \), find and simplify:

(a) \( f(t + 1) \)

(b) \( f(t^2 + 1) \)

(c) \( f(2) \)

(d) \( 2f(t) \)

(e) \( [f(t)]^2 + 1 \)

Solution (a): \( f(t + 1) = (t + 1)^2 + 1 = t^2 + 2t + 1 + 1 = t^2 + 2t + 2. \)

Solution (b): \( f(t^2 + 1) = (t^2 + 1)^2 + 1 = t^4 + 2t^2 + 2. \)

Solution (c): \( f(2) = 2^2 + 1 = 5. \)
Solution (d): \[2f(t) = 2(t^2 + 1) = 2t^2 + 2.\]

Solution (e): \[[f(t)]^2 + 1 = (t^2 + 1)^2 + 1 = t^4 + 2t^2 + 2.\]

1.8.8. Find the following if \(f(x) = 2x^2\) and \(g(x) = x + 3:\)

(a) \(f(g(x))\)
(b) \(g(f(x))\)
(c) \(f(f(x))\)

Solution (a): \(f(g(x)) = 2(x + 3)^2 = 2x^2 + 12x + 18.\)

Solution (b): \(g(f(x)) = 2x^2 + 3.\)

Solution (c): \(f(f(x)) = 2(2x^2)^2 = 8x^4.\)

1.8.14. Use the variable \(u\) for the inside function to express each of the following as a composite function:

(a) \(y = (5t^2 - 2)^6\)
(b) \(P = 12e^{-0.6t}\)
(c) \(C = 12\ln(q^3 + 1)\)

Solution (a): Set \(u = 5t^2 - 2\), so \(y = u^6.\)

Solution (b): Set \(u = -0.6t\), so \(P = 12e^u.\)

Solution (c): Set \(u = q^3 + 1\), so \(C = 12\ln u.\)

1.8.20. Estimate \(g(f(2))\) from the graphs of \(f\) and \(g.\)

Solution: First, from the graph of \(f\) above \(x = 2, f(2) \approx 0.4.\) Then from the graph of \(g\) above \(x = 0.4, g(0.4) \approx 1.2.\)
1.8.22. Using Table 1.36, create a table of values for \( f(g(x)) \) and for \( g(f(x)) \).

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>-2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0</td>
<td>-3</td>
</tr>
</tbody>
</table>

Solution: We calculate each value: for example, for \( f(g(-1)) \), we first look up \( g(-1) = -2 \). Then \( f(g(-1)) = f(-2) \), so we look that up to find \( f(-2) = 1 \). Hence, \( f(g(-1)) = 1 \).

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>( f(g(x)) )</th>
<th>( g(f(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>-2</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>1</td>
<td>-2</td>
</tr>
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<td>0</td>
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<td>-3</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

1.8.28. The Heaviside step function, \( H \), is graphed in Figure 1.79. Graph the following functions:

(a) \( 2H(x) \)
(b) \( H(x) + 1 \)
(c) \( H(x + 1) \)
(d) \( -H(x) \)
(e) \( H(-x) \)

Solution: Below are the graphs: