Homework #6 Solutions

Problems

Bolded problems are worth 2 points.

- Section 3.2: **4**, 16, 20, **26**, 34, **42**, 46
- Section 3.3: **4**, 12, 16, **26**, 34, **40**, 42, **50**

3.2.4. Differentiate the function $f(x) = x^3 + 3^x$.

Solution: Using the sum, power, and exponential rules, $f'(x) = 3x^2 + (\ln 3)3^x$. ∎

3.2.16. Differentiate $P = 200e^{0.12t}$.

Solution: The derivative is $P' = 200(0.12)e^{0.12t} = 24e^{0.12t}$. ∎

3.2.20. Differentiate $y = B + Ae^t$, where A and B are constants.

Solution: Using the constant-multiple, sum, and exponential rules, $y' = 0 + Ae^t = Ae^t$. ∎

3.2.26. Find the derivative of the function $R(q) = q^2 - 2\ln q$.

Solution: The derivative is $R'(q) = 2q - 2\frac{1}{q} = 2q - \frac{2}{q}$. ∎

3.2.34. The world's population is about $f(t) = 6.8e^{0.012t}$ billion, where t is the time in years since 2009. Find $f(0)$, $f'(0)$, $f(10)$, and $f'(10)$. Using units, interpret your answer in terms of population.

Solution: We find $f'(t) = 6.8(0.012)e^{0.012t} = 0.0816e^{0.012t}$, in billions of people per year. At $t = 0$, $f(0) = 6.8$ and $f'(0) = 0.0816$, so in 2009 the population is 6.8 billion and is increasing at a rate of 81.6 million per year. At $t = 10$, $f(10) = 7.67$ and $f'(10) = 0.092$, so the population in 2019 will be 7.67 billion and will be increasing at 92 million people per year. ∎

3.2.42. At a time t hours after it was administered, the concentration of a drug in the body is $f(t) = 27e^{-0.14t}$ ng/ml. What is the concentration 4 hours after it was administered? At what rate is the concentration changing at that time?

Solution: We find that $f'(t) = 27(-0.14)e^{-0.14t} = -3.78e^{-0.14t}$, in ng/ml · hr. At $t = 4$, the concentration is $f(4) = 27e^{-0.14(4)} \approx 15.42$ ng/ml, and the rate of change is $f'(4) = -3.78e^{-0.14(4)} \approx -2.16$ ng/ml · hr. ■

3.2.46. For the cost function $C = 1000 + 300 \ln q$ (in dollars), find the cost and the marginal cost at a production level of 500. Interpret your answers in economic terms.

Solution: The cost at $q = 500$ is $C(500) = 1000 + 300\ln(500) \approx 2864.38$ dollars. The marginal cost is the derivative of the cost, $C'(q) = \frac{300}{q}$, so at $q = 500$, $C'(500) = \frac{300}{500} = 0.60$ dollars per unit. Therefore, it costs \$2864.38 to make 500 units of this product, and at that level of production costs are increasing at a rate of \$0.60 per unit. ■

3.3.4. Find the derivative of the function $w = (t^2 + 1)^{100}$.

Solution: We use the chain rule: $w = f(z) = z^{100}$, where $z = g(t) = t^2 + 1$. Then $f'(z) = 100z^{99}$ and $g'(t) = 2t$, so

$$w' = 100z^{99}(2t) = 200t(t^2 + 1)^{99}.$$

■

3.3.12. Find the derivative of the function $w = e^{-3t^2}$.

Solution: We use the chain rule: first, we write $w = f(z) = e^z$, with $z = g(t) = -3t^2$. Then $f'(z) = e^z$ and $g'(t) = -6t$, so

$$w' = e^z(-6t) = -6te^{-3t^2}.$$

■

3.3.16. Find the derivative of $f(t) = \ln(t^2 + 1)$.

Solution: Using the chain rule, with $h(z) = \ln z$ the outer function and $z = g(t) = t^2 + 1$ the inner function, we have $h'(z) = \frac{1}{z}$ and $g'(t) = 2t$. Then

$$f'(t) = \frac{1}{z}(2t) = \frac{2t}{t^2 + 1}.$$

■

3.3.26. Find the derivative of $y = \sqrt{e^x + 1}$.

Solution: We write this function as a composite: $y = \sqrt{z}$, where $z = e^x + 1$. Since $y = \sqrt{z} = z^{1/2}$, we use the power rule to find its derivative as $\frac{1}{2}z^{-1/2}$. $z' = e^x$, so the overall derivative is

$$y' = \frac{1}{2}(e^x + 1)^{-1/2}(e^x) = \frac{e^x}{2\sqrt{e^x + 1}}.$$

■

3.3.34. Find the relative rate of change $\frac{f'(t)}{f(t)}$ of the function $f(t) = 35t^{-4}$.

Solution: Since $f'(t) = 35(-4)t^{-5}$, the relative rate of change is

$$\frac{f'(t)}{f(t)} = \frac{35(-4)t^{-5}}{35t^{-4}} = -4t^{-1} = -\frac{4}{t}.$$

■

3.3.40. A firm estimates that the total revenue, R, received from the sale of q goods is given by

$$R = \ln(1 + 1000q^2).$$

Calculate the marginal revenue when $q = 10$.

Solution: The derivative of the revenue function, $R'(q)$, gives the marginal revenue. This derivative is

$$R'(q) = \frac{2000q}{1 + 1000q^2}.$$

At $q = 10$, $R'(10) = \frac{2000(10)}{1 + 1000(10)^2} = \frac{20{,}000}{100{,}001} \approx 0.20.$

■

3.3.42. If you invest P dollars in a bank account at an annual interest rate of r%, then after t years you will have B dollars, where

$$B = P\left(1 + \frac{r}{100}\right)^t.$$

(a) Find $\frac{dB}{dt}$, assuming P and r are constant. In terms of money, what does $\frac{dB}{dt}$ represent?

(b) Find $\frac{dB}{dr}$, assuming P and t are constant. In terms of money, what does $\frac{dB}{dr}$ represent?

Solution (a): With t as the independent variable, we recognize $B(t) = P\left(1 + \frac{r}{100}\right)^t$ as an exponential function Pa^t with base $a = 1 + \frac{r}{100}$. Therefore, its derivative is

$$\frac{dB}{dt} = P(\ln a)a^t = P\left(\ln\left(1 + \frac{r}{100}\right)\right)\left(1 + \frac{r}{100}\right)^t.$$

This derivative tells us how fast the balance at a fixed rate r changes over time, in units of dollars per year. ■

Solution (b): With r as the independent variable, we see that $B(r) = P\left(1 + \frac{r}{100}\right)^t$ is more like a power function. Let $z = g(r) = 1 + \frac{r}{100}$, and then $B = Pz^t$, where t is constant. Hence, since $z' = \frac{1}{100}$,

$$\frac{dB}{dr} = Ptz^{t-1}\frac{1}{100} = \frac{Pt}{100}\left(1 + \frac{r}{100}\right)^{t-1}.$$

This derivative tells us how fast the balance changes as we change the interest rate, r, but let the interest accumulate over the same period of time, t. Its units are in dollars per percentage point. ■

3.3.50. Let $h(x) = f(g(x))$, where f and g are graphed as in the text. Estimate $h'(2)$.

Solution: By the chain rule, $h'(2) = f'(g(2))g'(2)$. First, we estimate that $g(2) \approx 1.6$, so $h'(2) = f'(1.6)g'(2)$. Next, from the slopes of tangent lines to the given graphs, we estimate that $g'(2) \approx -2$ and $f'(1.6) \approx 1$, so $h'(2) = (-2)(1) = -2$. Note that these derivative estimates are difficult to make and so answers may vary substantially. ■