Extra-Credit Take-Home Quiz: Due Tue, Dec 6

Name: Solution Key Recitation: R02 R03 R04

- You may consult your textbook, notes, and course materials, and you may use a calculator or a computer program for computations.
- Please work on the problems on your own and do not discuss them with other people. You may ask the instructor or TAs for help or clarifications on the problems, however.

1. Let $f(x) = \frac{1}{3}x^3 - 3x^2 + 5x + 10$.

(a) Find $f'(x)$, and use it to find the critical points of $f(x)$.

Solution: $f'(x) = x^2 - 6x + 5$. We set $f'(x) = 0$ and solve for x, so $x^2 - 6x + 5 = 0$. The left-hand side factors as

$$(x-1)(x-5) = 0,$$

so the solutions are $x = 1$ and $x = 5$. These are the critical points.

(b) Find the global maximum and minimum values of $f(x)$ on the interval $[0, 9]$ and the x-values where they occur.

Solution: We evaluate $f(x)$ at the critical points $x = 1$ and $x = 5$ and at the endpoints $x = 0$ and $x = 9$:

$$\begin{aligned}
f(0) &= \frac{1}{3}(0)^3 - 3(0)^2 + 5(0) + 10 = 10, \\
f(1) &= \frac{1}{3}(1)^3 - 3(1)^2 + 5(1) + 10 = \frac{1}{3} + 12 = 12.333\ldots, \\
f(5) &= \frac{1}{3}(5)^3 - 3(5)^2 + 5(5) + 10 = -\frac{25}{3} + 10 = \frac{5}{3} = 1.666\ldots, \\
f(9) &= \frac{1}{3}(9)^3 - 3(9)^2 + 5(9) + 10 = 55.
\end{aligned}$$

The highest of these values is 55, so that is the global maximum value of $f(x)$ on this interval, and it occurs at $x = 9$. The lowest of the four values is $\frac{5}{3}$, so this is the global minimum value on this interval, and it occurs at $x = 5$.

2. We sell cups of hot chocolate on a cold winter's day. We know that to sell q cups in a day, we must set the price at $p(q) = 4 - \frac{1}{2000}q$ dollars.

(a) Use this price function to find the revenue function $R(q)$ in terms of q.

Solution: The revenue $R(q)$ is given by the price, $p(q)$, times the quantity, q, so is

$$R(q) = qp(q) = q\left(4 - \frac{1}{2000}q\right) = 4q - \frac{1}{2000}q^2.$$

(b) Find the marginal revenue in terms of q. What value of q maximizes the revenue? What are the price and the revenue at this q?

Solution: Taking the derivative of $R(q)$ gives the marginal revenue,

$$R'(q) = 4 - \frac{1}{2000}(2q) = 4 - \frac{1}{1000}q.$$

We set this equal to 0 and solve for q: $4 - \frac{1}{1000}q = 0$, so $q = 4(1000) = 4000$. To sell 4000 cups, the price should be $p(4000) = 4 - \frac{4000}{2000} = 4 - 2 = 2$, so the revenue is $2 \cdot 4000 = 8000$ dollars.

To be profitable, we also need to take into account our costs. It costs us \$200 per day to operate plus \$2 per cup of hot chocolate, so the cost function is $C(q) = 200 + 2q$.

(c) Find the value of q that maximizes our profit. What is the price at that q, and what profit do we make?

Solution: To maximize the profits, we set $R'(q) = C'(q)$ and solve for q. We computed $R'(q)$ above, and $C'(q) = (200 + 2q)' = 2$, so this equation is

$$4 - \frac{1}{1000}q = 2.$$

Then $q = 1000(4 - 2) = 2000$ will yield the maximum profit. At this quantity, the price should be $p(2000) = 4 - \frac{2000}{2000} = 4 - 1 = 3$ dollars. The revenue is $R = pq = 3(2000) = 6000$, and the cost is $C(2000) = 200 + 2(2000) = 4200$, so the profit is $6000 - 4200 = 1800$ dollars.