Midterm #2 — November 8, 2011, 8:30 to 10:00 PM

Name: Solution Key (A)

Circle your recitation:

R02 (Chira, Mon) R03 (Chira, Wed) R04 (Marcelo, Tue)

- **You have a maximum of $1\frac{1}{2}$ hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids are allowed.
- Read each question carefully. Show your work and justify your answers for full credit. You do not need to simplify your answers unless instructed to do so. Circle, box, or otherwise point out your final answer if it is not obvious.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do *not* unstaple or detach pages from this exam.

Grading

1	/30
2	/15
3	/15
4	/15
5	/15
6	/10
Total	/100

1. (*30 points*) Find the derivative of each function below. Simplify your answers.

(a) (*5 points*) $f(x) = x^5 - 3x + 7$

Solution: Using the power rule, the constant multiple rule, and the sum rule,

$$f'(x) = 5x^4 - 3.$$

(b) (*5 points*) $H(x) = (x^5 - 2x)^3$

Solution: This function is a composite, so we write $z = g(x) = x^5 - 2x$ as the inner function and $f(z) = z^3$ as the outer function. Then $f'(z) = 3z^2$ and $g'(x) = 5x^4 - 2$, so

$$H'(x) = f'(z)g'(x) = 3(x^5 - 2x)^2(5x^4 - 2).$$

(c) (*5 points*) $J(z) = \dfrac{z}{2 - z^2}$

Solution: This function is a quoitent of $f(z) = z$ and $g(z) = 2 - z^2$, so we use the quotient rule. $f'(z) = 1$ and $g'(z) = -2z$, so

$$J'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2} = \frac{1(2 - z^2) - z(2z)}{(2 - z^2)^2}$$

Simplifying,

$$J'(z) = \frac{2 - z^2 + 2z^2}{(2 - z^2)^2} = \frac{2 + z^2}{(2 - z^2)^2}.$$

1., continued.

(d) (*5 points*) $h(t) = (t^2 - 2t + 2)e^t$

Solution: This function is the product of two functions, $f(t) = t^2 - 2t + 2$ and $g(t) = e^t$. Then $f'(t) = 2t - 2$ and $g'(t) = e^t$, so by the product rule,

$$h'(t) = f'(t)g(t) + f(t)g'(t) = (2t - 2)e^t + (t^2 - 2t + 2)e^t.$$

Factoring out the e^t, we have

$$h'(t) = (2t - 2 + t^2 - 2t + 2)e^t = t^2 e^t.$$

(e) (*5 points*) $A(u) = 6\sqrt[3]{u} + \frac{1}{u^4} - 2\ln u$

Solution: Rewriting $A(u)$ with exponents, $A(u) = 6u^{1/3} + u^{-4} - 2\ln u$. Taking derivatives,

$$A'(u) = 6 \cdot \frac{1}{3}u^{-2/3} - 4u^{-5} - \frac{2}{u} = 2u^{-2/3} - 4u^{-5} - \frac{2}{u}.$$

(f) (*5 points*) $V(t) = \frac{te^{t^2-3}}{t+5}$

Solution: $V(t)$ is the quotient of $f(t) = te^{t^2-3}$ by $g(t) = t + 5$. We use the product and chain rules to find

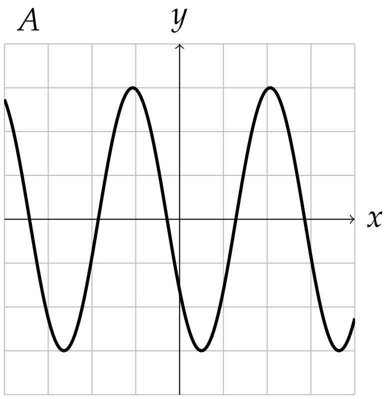
$$f'(t) = (1)e^{t^2-3} + t(2t)e^{t^2-3} = (1 + 2t^2)e^{t^2-3}.$$

Then using the quotient rule with $g'(t) = 1$ and simplifying the numerator,

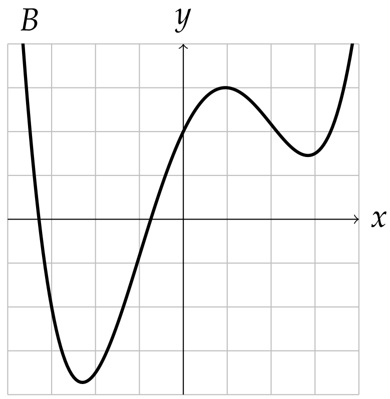
$$V'(t) = \frac{(1 + 2t^2)e^{t^2-3}(t+5) - te^{t^2-3}}{(t+5)} = \frac{(2t^3 + 10t^2 + 5)e^{t^2-3}}{(t+5)^2}$$

2. (*15 points*) Below are the graphs of six functions, labeled A through F:

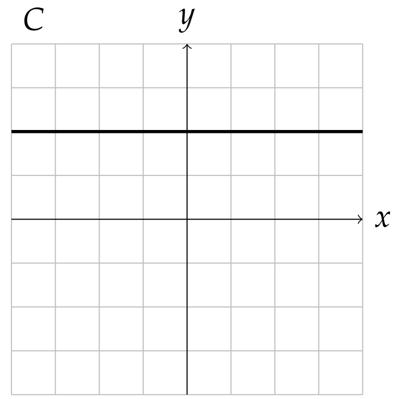
A



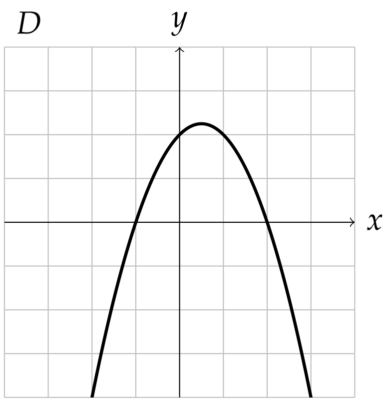
B



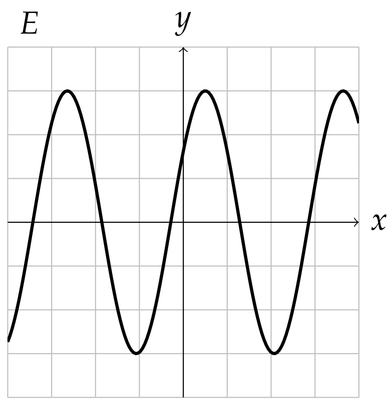
C



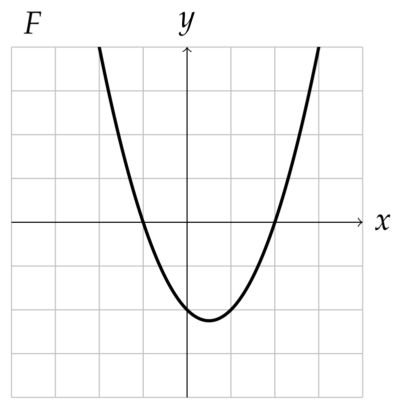
D



E

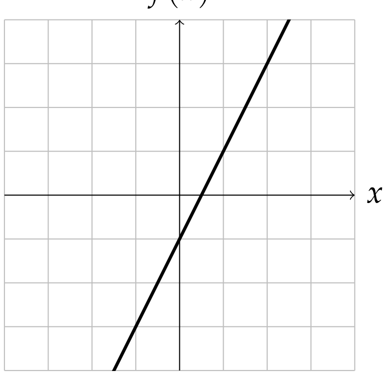


F

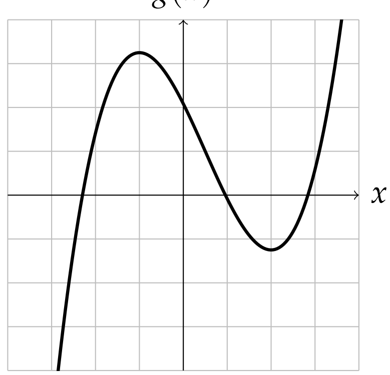


We also have graphs of three functions, $f(x)$, $g(x)$, and $h(x)$. Identify which of the graphs A–F are the graphs of their derivatives, $f'(x)$, $g'(x)$, and $h'(x)$.

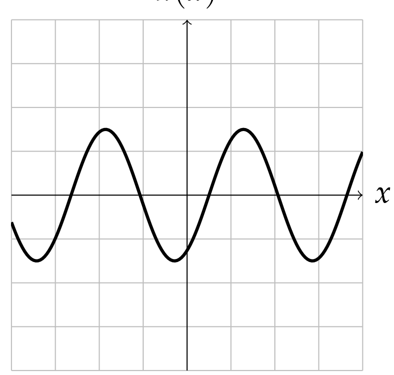
$f(x)$



$g(x)$



$h(x)$



$f'(x)$: ___C___ $g'(x)$: ___F___ $h'(x)$: ___E___

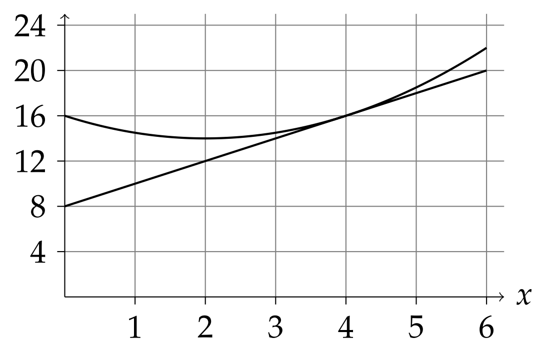
3. (*15 points*) The function $P(x)$ gives the population of Springfield, in thousands of people, where x is in years since 2005. The tangent line to the function $P(x)$ at $x = 4$ is given by $y = 8 + 2x$.

(a) (*4 points*) Find the values of $P(4)$ and $P'(4)$. What do these numbers mean? Include units in your explanation.

Solution: The tangent line provides the both the value of $P(x)$ and its slope $P'(x)$ at the basepoint $x = 4$, so $P(4) = 8 + 2(4) = 16$, and $P'(4) = 2$. Hence, in 2009, the town has a population of 16,000 and is increasing at a rate of 2,000 per year.

(b) (*4 points*) Suppose that $P''(4)$ is positive. On the axes to the right, sketch graphs of both $P(x)$ and the tangent line that are consistent with the above information.

Solution: The tangent line is given above. $P(x)$ can vary greatly, but should touch the tangent line at $x = 4$ and should have the correct concavity there.



(c) (*4 points*) Estimate $P(3.5)$.

Solution: Using the tangent line, we estimate $P(3.5)$ to be $8 + 2(3.5) = 15$.

(d) (*3 points*) Is your estimate in part (c) an underestimate or an overestimate? Explain.

Solution:

Solution: Since $P''(4)$ is positive, the tangent line lies below the actual $P(x)$ curve, so the estimate is an underestimate.

4. (*15 points*) Let $f(x) = \dfrac{x^3}{3} - 3x^2 - 7x + 3$.

(a) (*3 points*) Find $f'(x)$. Simplify your answer.

Solution: $f'(x) = x^2 - 6x - 7$.

(b) (*2 points*) Find $f''(x)$. Simplify your answer.

Solution: $f''(x) = 2x - 6$.

(c) (*3 points*) Find the critical points of $f(x)$.

Solution: We set $f'(x) = 0$ and solve for x to find the critical points. Then $x^2 - 6x - 7 = 0$, which factors as

$$(x-7)(x+1) = 0.$$

The solutions are $x = 7$ and $x = -1$, so these are the critical points.

4., continued.

(d) (*3 points*) For each critical point of $f(x)$, decide whether it is a local maximum, a local minimum, or neither. Explain your answers.

Solution: Since we computed the second derivative in part (b), we try the second derivative test. We evaluate $f''(x) = 2x - 6$ at each critical point and check the sign:

$$f''(7) = 8 > 0, \qquad f''(-1) = -8 < 0.$$

Therefore, $f(x)$ has a local minimum at $x = 7$ and a local maximum at $x = -1$.

(e) (*2 points*) On which x-intervals is $f(x)$ increasing? Decreasing?

Solution: The critical points divide the real line, the domain of f, into three intervals, $(-\infty, -1)$, $(-1, 7)$, and $(7, \infty)$. Since $f(x)$ has a local maximum at $x = -1$, $f(x)$ must be increasing on $(-\infty, -1)$ and decreasing on $(-1, 7)$. Since $f(x)$ has a local minimum at $x = 7$, $f(x)$ must also be increasing on $(7, \infty)$. Then

- $f(x)$ is increasing on $(-\infty, -1)$ and $(7, \infty)$, and
- $f(x)$ is decreasing on $(-1, 7)$.

(f) (*2 points*) Find the inflection points of $f(x)$. Justify your answers.

Solution: We look for inflection points where $f''(x) = 0$. Then $2x - 6 = 0$, so $x = 3$. From the calculations of part (d), we see that the concavity changes from $-$ to $+$ across this point, so there is indeed an inflection point there.

5. (*15 points*) Below are the values of a function $w(t)$ at regularly spaced points between $t = 1$ and $t = 2$:

t	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$w(t)$	2.44	2.54	2.66	2.82	3	3.24	3.54	3.9	4.34	4.8	5.3

(a) (*8 points*) Estimate $w'(1.2)$ and $w'(1.7)$. Explain your estimates.

Solution: We take the average of $w(t)$ over small intervals centered about 1.2 and about 1.7, so from 1.1 to 1.3 and from 1.6 to 1.8:

$$w'(1.2) \approx \frac{w(1.3) - w'(1.1)}{1.3 - 1.1} = \frac{2.82 - 2.54}{0.2} = \frac{0.28}{0.2} = 1.4,$$
$$w'(1.7) \approx \frac{w(1.8) - w'(1.6)}{1.8 - 1.6} = \frac{4.34 - 3.54}{0.2} = \frac{0.8}{0.2} = 4.$$

(b) (*4 points*) Do you think $w''(t)$ is positive, negative, or neither between $t = 1$ and $t = 2$? Why?

Solution: Since $w'(t)$ increases from 1.4 to 4 as t increases from 1.2 to 1.7, we expect that $w''(t)$ is positive on this interval.

(c) (*3 points*) Estimate $w''(1.4)$. Explain how you computed your estimate.

Solution: We first estimate $w'(1.35)$ and $w'(1.45)$ over the intervals from 1.3 to 1.4 and from 1.4 to 1.5:

$$w'(1.35) \approx \frac{w(1.4) - w(1.3)}{1.4 - 1.3} = \frac{3 - 2.82}{0.1} = \frac{0.18}{0.1} = 1.8,$$
$$w'(1.45) \approx \frac{w(1.5) - w(1.4)}{1.5 - 1.4} = \frac{3.24 - 3}{0.1} = \frac{0.24}{0.1} = 2.4.$$

We then estimate w'', the rate of change of w', from these two values:

$$w''(1.4) \approx \frac{w'(1.45) - w'(1.35)}{1.45 - 1.35} = \frac{2.4 - 1.8}{0.1} = \frac{0.6}{0.1} = 6.$$

6. (*10 points*) We sell tickets to a soccer game. We determine that if we set the price to be p dollars, the number of tickets we will sell is $q(p) = 100e^{(1-p/20)}$.

(a) (*3 points*) Write a formula for the total revenue $R(p)$ we collect, in terms of the price p.

Solution: The revenue is the price p times the quantity $q(p)$, so

$$R(p) = pq(p) = p(100e^{(1-p/20)}) = 100pe^{(1-p/20)}.$$

(b) (*4 points*) Find the price p giving the maximum revenue and the revenue at this price.

Solution: To find the maximum of $R(p)$, we compute its derivative and solve $R'(p) = 0$. We use the product rule and the chain rule:

$$R'(p) = 100\left((1)e^{(1-p/20)} + p\left(-\frac{1}{20}\right)e^{(1-p/20)}\right) = 100\left(1 - \frac{p}{20}\right)e^{(1-p/20)}.$$

Since the leading 100 and the exponential are both never 0, the only solutions come from $1 - \frac{p}{20} = 0$. Then $p = 20$ is the only critical point. In fact, since $R'(p)$ is positive to the left of $p = 20$ and negative to the right, this is a local (and in fact global) maximum.

The revenue at this price is

$$R(20) = 100(20)e^{(1-20/20)} = 2000e^0 = 2000.$$

(c) (*3 points*) Find the price p at which the revenue is decreasing the fastest.

Solution: The price is decreasing the fastest when $R'(p)$ is at a minimum, which happens at a critical point of $R'(p)$. Therefore, we compute $R''(p)$ and set it to 0:

$$\begin{aligned} R''(p) &= 100\left[\left(-\frac{1}{20}\right)e^{(1-p/20)} + \left(1 - \frac{p}{20}\right)\left(-\frac{1}{20}\right)e^{(1-p/20)}\right] \\ &= 100\left(\frac{p}{20^2} - \frac{2}{20}\right)e^{(1-p/20)} \end{aligned}$$

Then $R''(p)$ is 0 only when $p = 20 \cdot 2 = 40$. Since $R''(p)$ is negative to the left of $p = 40$ and positive to the right, this critical point of $R'(p)$ is indeed a local minimum. Note that this point is also an inflection point of the original revenue function $R(p)$.