Homework #12 Solutions

Problems

Bolded problems are worth 2 points.

- Section 5.3: 2, 6, **12**, **18**, **20**, 26
- Section 6.1: 4, **10**, 12, **16**, **20**

5.3.2. Find the area under $P = 100(0.6)^t$ between $t = 0$ and $t = 8$.

Solution: Since P is an exponential function, it is always positive, so the graph of P lies above the t-axis. Hence, the integral

$$\int_0^8 100(0.6)^t\,dt$$

is this area. We find an antiderivative $G(t)$ for P, which is $G(t) = \frac{100}{\ln(0.6)}(0.6)^t$. Then

$$\int_0^8 100(0.6)^t\,dt = \frac{100}{\ln(0.6)}(0.6)^t\Big|_0^8 = \frac{100}{\ln(0.6)}\left((0.6)^8 - (0.6)^0\right) \approx 192.47. \qquad \blacksquare$$

5.3.6. From the graph, decide whether $\int_{-3}^{3} f(x)\,dx$ is positive, negative, or approximately zero.

Solution: From the graph, there is a lot of area below the x-axis and comparatively little above the x-axis, so overall the integral will be negative. ■

5.3.12. Given $\int_{-1}^{0} f(x)\,dx = 0.25$ and the figure, estimate:

(a) $\int_0^1 f(x)\,dx$

(b) $\int_{-1}^1 f(x)\,dx$

(c) The total shaded area.

Solution (a): The area enclosed by the graph from $x = 0$ to $x = 1$ is the same as that enclosed from $x = -1$ to $x = 0$, but is below the x-axis, so the integral $\int_0^1 f(x)\,dx = -0.25$. ■

Solution (b): We add the two integrals to get $0.25 + (-0.25) = 0$. ■

Solution (c): The total area is the sum of the area of the two separate regions, which is $0.25 + 0.25 = 0.5$. ■

5.3.18. Use Figure 5.39 to find the values of
(a) $\int_a^b f(x)\,dx$
(b) $\int_b^c f(x)\,dx$
(c) $\int_a^c f(x)\,dx$
(d) $\int_a^c |f(x)|\,dx$

Solution (a): Since $f(x)$ is positive between a and b, the integral $\int_a^b f(x)\,dx$ is the same as the area enclosed by the graph on that interval, and so is 13. ■

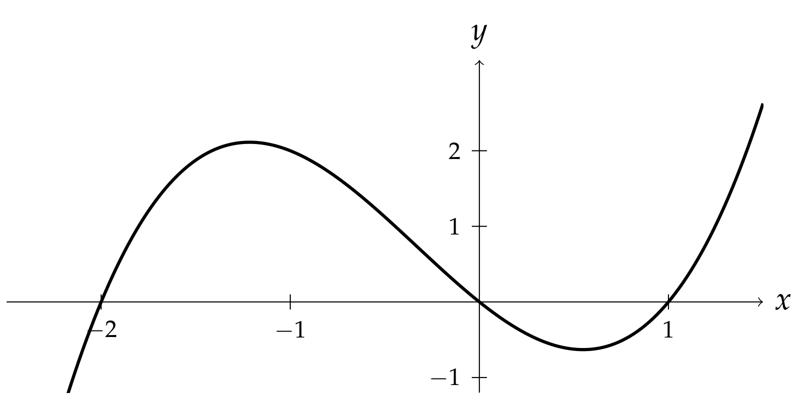
Solution (b): Since $f(x)$ is negative between b and c, the integral $\int_b^c f(x)\,dx$ is the negative of the area enclosed by the graph on that interval, and so is -2. ■

Solution (c): The integral $\int_a^c f(x)\,dx$ is the sum of the integrals $\int_a^b f(x)\,dx$ and $\int_b^c f(x)\,dx$, so is $13 + (-2) = 11$. ■

Solution (d): For the integral of the absolute value of f, all the areas become positive, so the integral $\int_a^c |f(x)|\,dx$ is $13 + 2 = 15$. ■

5.3.20. Let $f(x) = x(x+2)(x-1)$.
(a) Graph $f(x)$.
(b) Find the total area between the graph and the x-axis between $x = -2$ and $x = 1$.
(c) Find $\int_{-2}^{1} f(x)\,dx$ and interpret it in terms of areas.

Solution (a): A graph of $f(x) = x(x+2)(x-1)$ is given below:



■

Solution (b): The region enclosed by the graph and the x-axis is formed by the region above the x-axis between $x = -2$ and $x = 0$ and the region below the x-axis between $x = 0$ and $x = 1$.

- The area of the first region is given by the integral $\int_{-2}^{0} f(x)\,dx$. We compute an antiderivative of $f(x) = x(x+2)(x-1) = x^3 + x^2 - 2x$, which is $G(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2$. Then

$$\int_{-2}^{0} f(x)\,dx = \frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2 \Big|_{-2}^{0} = 0 - (\frac{16}{4} + \frac{-8}{3} - 4) = \frac{8}{3}.$$

- The integral $\int_0^1 f(x)\,dx$ is

$$\int_{0}^{1} f(x)\,dx = \frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2 \Big|_{0}^{1} = \frac{1}{4} + \frac{1}{3} - 1 = -\frac{5}{12}.$$

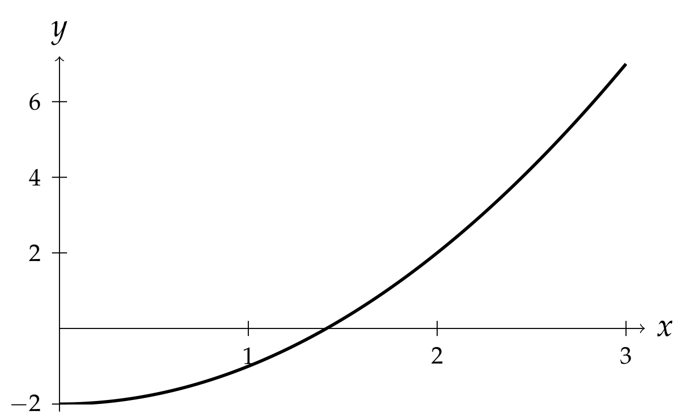
  Therefore, the area of the second region is $\frac{5}{12}$, changing the sign on the negative number given by the integral.

The total area is then $\frac{8}{3} + \frac{5}{12} = \frac{37}{12} \approx 3.08333\ldots$. ■

Solution (c): The integral $\int_{-2}^{1} f(x)\,dx$ is the sum of the two integrals computed above, so is $\frac{8}{3} - \frac{5}{12} = \frac{9}{4} = 2.25$. This is the difference between the areas of the two regions. ■

5.3.26. Find the area between the graph of $y = x^2 - 2$ and the x-axis between $x = 0$ and $x = 3$.

Solution: We graph the function $f(x) = x^2 - 2$ from 0 to 3:



So the graph determines two regions, one below the x-axis and one above. The point between the two of them is determined by $x^2 - 2 = 0$, so $x^2 = 2$, and $x = \sqrt{2}$. We therefore compute two separate integrals, one from 0 to $\sqrt{2}$ and another from $\sqrt{2}$ to 3, and change the value of the first one to be positive.

- For the first integral, we find an antiderivative $G(x)$ for $f(x) = x^2 - 2$, which is $G(x) = \frac{1}{3}x^2 - 2x$. Then

$$\int_0^{\sqrt{2}} x^2 - 2\,dx = \frac{1}{3}x^2 - 2x\Big|_0^{\sqrt{2}} = \frac{2\sqrt{2}}{3} - 2\sqrt{2} - (0-0) = -\frac{4\sqrt{2}}{3}.$$

  The area is then $\frac{4\sqrt{2}}{3}$.
- For the second integral,

$$\int_{\sqrt{2}}^{3} x^2 - 2\,dx = \frac{1}{3}x^2 - 2x\Big|_{\sqrt{2}}^{3} = (9-6) - \left(-\frac{4\sqrt{2}}{3}\right) = 3 + \frac{4\sqrt{2}}{3}.$$

Therefore, the total area is $3 + \frac{4\sqrt{2}}{3} + \frac{4\sqrt{2}}{3} = 3 + \frac{8\sqrt{2}}{3} \approx 6.771$. ■

6.1.4. Find the average value of the function $f(x) = 5 + 4x - x^2$ between $x = 0$ and $x = 3$.

Solution: This average is given by

$$\frac{1}{3-0}\int_0^3 5 + 4x - x^2\,dx = \frac{1}{3}\left[5x + 2x^2 - \frac{1}{3}x^3\right]_3^0 = \frac{1}{3}[(15+18-9) - 0] = 8.$$ ■

6.1.10. The annual income for ages 25 to 85 is given graphically.

(a) Find the average annual income for these years.

(b) Assuming that the person spends at a constant rate equal to their average income, when are they spending less than they earn, and when are they spending more?

Solution (a): We compute the integral of the annual income function from the area on the graph. The only area is contributed by the region from $t = 25$ to $t = 65$. The annual income is given by $A(t) = 2t - 30$, in thousands of dollars, so the region has area

$$\int_{25}^{65} 2t - 30\,dt = t^2 - 30t\Big|_{25}^{65} = (65^2 - 30(65)) - (25^2 - 30(25)) = 2400.$$

Averaging this from $t = 25$ to $t = 85$, we get

$$\frac{2400}{85-25} = \frac{2400}{60} = 40.$$

Hence, the average annual income is $40,000 per year. ■

Solution (b): From $t = 25$ to $t = 35$ and from $t = 65$ to $t = 85$, the actual income is less than the average, so the person is spending more than they make. From $t = 35$ to $t = 65$, the actual income is above the average, so the person is spending less than they make. ■

6.1.12. If t is measured in days since June 1, the inventory $I(t)$ for an item in a warehouse is given by $I(t) = 5000(0.9)^t$.

(a) Find the average inventory in the warehouse during the 90 days after June 1.

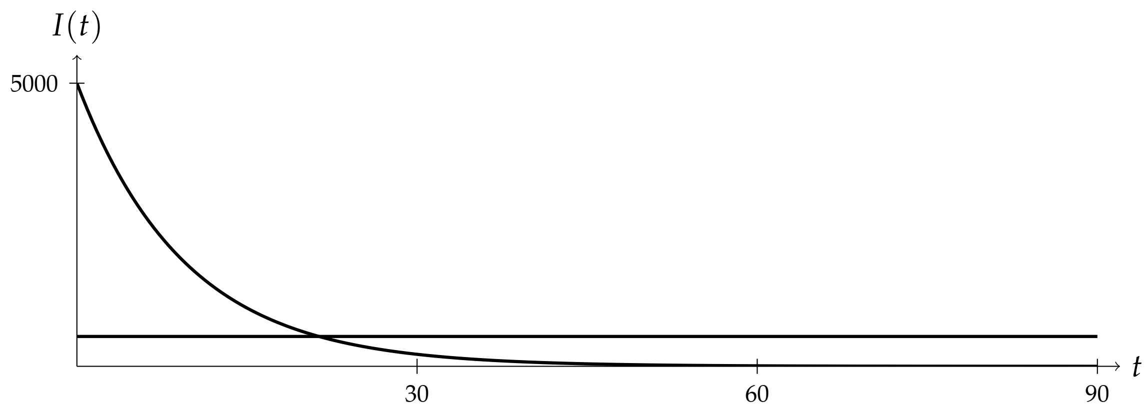
(b) Graph $I(t)$ and illustrate the average graphically.

Solution (a): The average inventory is given by

$$\frac{1}{90}\int_0^{90} 5000(0.9)^t\,dt = \frac{1}{90}\left[\frac{5000}{\ln 0.9}(0.9)^t\right]_0^{90} = \frac{5000}{90\ln 0.9}\left((0.9)^{90}-1\right) \approx 527.25.$$

■

Solution (b): We graph both $I(t) = 5000(0.9)^t$ and the average 527.25:



■

6.1.16. The population of the world t years after 2000 is predicted to be $P = 6.1e^{0.0125t}$ billion.

(a) What population is predicted in 2010?

(b) What is the predicted average population between 2000 and 2010?

Solution (a): The population in 2010 is $P(10) = 6.1e^{0.0125(10)} = 6.1e^{0.125} \approx 6.91$ billion. ■

Solution (b): The average population is, in billions,

$$\frac{1}{10}\int_0^{10} 6.1e^{0.0125t}\,dt = \frac{1}{10}\left[\frac{6.1}{0.0125}e^{0.0125t}\right]_0^{10} = 48.8(e^{0.125}-1) \approx 6.498.$$

■

6.1.20. Throughout much of the 20th century, the yearly consumption of electricity in the US increased exponentially at a continuous rate of 7% per year. Assume this trend continues and that the electrical energy consumed in 1900 was 1.4 million megawatt-hours.

(a) Write an expression for yearly electricity consumption as a function of time, t, in years since 1900.

(b) Find the average yearly electrical consumption throughout the 20th century.

(c) During what year was electrical consumption closest to the average for the century?

(d) Without doing the calculation for part (c), how could you have predicted which half of the century the answer would be in?

Solution (a): The yearly consumption is given, in million megawatt-hours, by the exponential function

$$f(t) = 1.4e^{0.07t}.$$

■

Solution (b): The average yearly consumption is given by

$$\frac{1}{100}\int_0^{100} 1.4e^{0.07t}\,dt = \frac{1}{100}\left[\frac{1.4}{0.07}e^{0.07t}\right]_0^{100} = \frac{1}{5}(e^7 - 1) \approx 219.12.$$

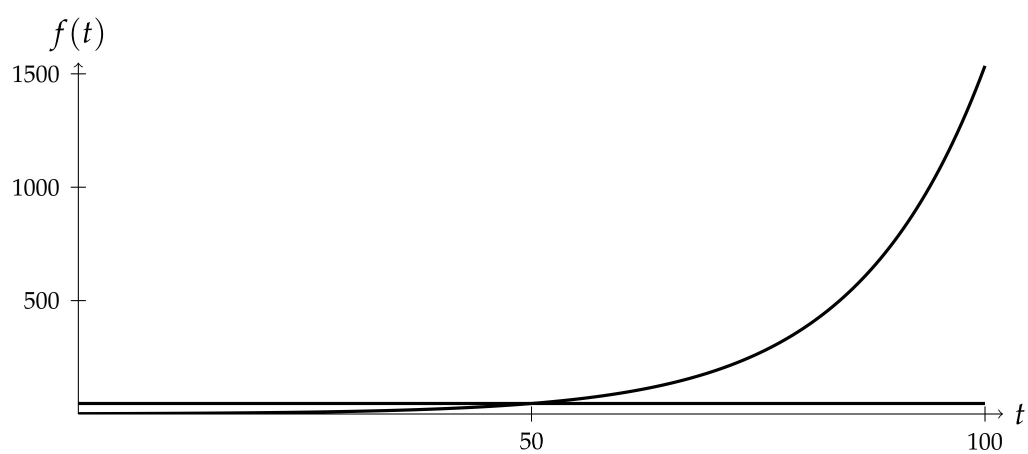
■

Solution (c): We solve $1.4e^{0.07t} = 219.12$. Dividing, $e^{0.07t} = 219.12/1.4$, so

$$t = \frac{1}{0.07}\ln\frac{219.12}{1.4} \approx 72.2.$$

Thus, the consumption is closest to the average in 1972. ■

Solution (d): Since the growth is exponential, the rate grows faster in the second half of the century than it does in the first half. Hence, if we compare the consumption function $f(t)$ to the value at $t = 50$, the "missing" area below $y = f(50)$ and above $y = f(t)$ is far less than the "excess" area below $y = f(t)$ and above $y = f(50)$, so $f(50)$ is too small to be the average:



Therefore, the average is higher, so since $f(t)$ is increasing, it meets the $f(t)$ graph to the right of $t = 50$. ■