Solutions to Final Practice Problems

1. Find the general solution, possibly implicit, to each of the following DEs or systems of DEs. If an initial condition is given, also find the particular solution matching it.

(a) $x' = 3x + 4y$, $y' = 3x + 2y$, $x(0) = 1$, $y(0) = 1$

Solution: We write this system as $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$, and compute the eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 6 = (\lambda - 6)(\lambda + 1).$$

Therefore, the eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = -1$. We then find eigenvectors for each eigenvalue. For $\lambda_1 = 6$, we row reduce $A - 6I$:

$$A - 6I = \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 \\ 0 & 0 \end{bmatrix}.$$

Then an eigenvector for $\lambda_1 = 6$ is $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Repeating this process for $\lambda_2 = -1$, we row reduce

$$A + I = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

so $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a choice of eigenvector for $\lambda_2 = -1$. Hence, the general solution is

$$\mathbf{x}(t) = c_1 e^{6t} \begin{bmatrix} 4 \\ 3 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We also match the initial condition, which we write in vector format as $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

$$c_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which we solve by converting into augmented matrix format and row reducing:

$$\left[\begin{array}{cc|c} 4 & 1 & 1 \\ 3 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 1 & 1 \\ 0 & -7 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 0 & \frac{8}{7} \\ 0 & 1 & -\frac{1}{7} \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{2}{7} \\ 0 & 1 & -\frac{1}{7} \end{array}\right]$$

Then the solution is

$$\mathbf{x}(t) = \frac{2}{7} e^{6t} \begin{bmatrix} 4 \\ 3 \end{bmatrix} - \frac{1}{7} e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 8e^{6t} - e^{-t} \\ 6e^{6t} + e^{-t} \end{bmatrix}.$$

(b) $\left(x^3 + \frac{y}{x}\right) dx + (y^2 + \ln x)\, dy = 0$

Solution: Because of the format of the DE and the nonlinearity, we check this DE for exactness, with $M(x,y) = x^3 + \frac{y}{x}$ and $N(x,y) = y^2 + \ln x$. Then $M_y = \frac{1}{x}$, and $N_x = \frac{1}{x}$, so the DE is exact. We integrate M with respect to x:

$$F(x,y) = \int M\, dx = \int x^3 + \frac{y}{x}\, dx = \frac{1}{4}x^4 + y \ln x + g(y),$$

where $g(y)$ is an unknown function of y alone. Differentiating with respect to y and comparing the result to N, we have

$$F_y(x,y) = \ln x + g'(y) = y^2 + \ln x,$$

so $g'(y) = y^2$. Hence, one antiderivative is $g(y) = \frac{1}{3}y^3$, so the solutions are defined implicitly by

$$F(x,y) = \frac{1}{4}x^4 + y \ln x + \frac{1}{3}y^3 = C.$$

(c) $x_1' = -x_1 - 3x_2 + 2x_3$, $x_2' = x_1 + 2x_2 - x_3$, $x_3' = -x_1 - 2x_2 + 2x_3$

Solution: We write this system as $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} -1 & -3 & 2 \\ 1 & 2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$, and compute the eigenvalues of A:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & -3 & 2 \\ 1 & 2-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{vmatrix} \\
&= -(1+\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ -1 & 2-\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & 2-\lambda \\ -1 & -2 \end{vmatrix} \\
&= -(1+\lambda)((2-\lambda)^2 - 2) + 3(2 - \lambda - 1) + 2(-2 + 2 - \lambda) \\
&= -(1+\lambda)(\lambda^2 - 4\lambda + 2) + 3 - 3\lambda - 2\lambda \\
&= -(\lambda^3 - 4\lambda^2 + 2\lambda + \lambda^2 - 4\lambda + 2) + 3 - 5\lambda \\
&= -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3.
\end{aligned}$$

Then the only eigenvalue is $\lambda = 1$, with multiplicity 3. We compute its eigenvectors $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ by row reducing $A - I$:

$$A - I = \begin{bmatrix} -2 & -3 & 2 \\ 1 & 1 & -1 \\ -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $a = c$ and $b = 0$, so $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is the only linearly independent eigenvector for this eigenvalue. Therefore, we must look for two more generalized eigenvectors in a tower over this one. For the first of those, we seek a vector $\mathbf{v}_2$ so that $(A - I)\mathbf{v}_2 = \mathbf{v}_1$. To solve this, we row reduce the augmented system $[A - I \mid \mathbf{v}_1]$:

$$\left[\begin{array}{ccc|c} -2 & -3 & 2 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & -2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then we may take $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (although choices differening by a multiple of $\mathbf{v}_1$ are also valid). Finally, we must also find a vector $\mathbf{v}_3$ so that $(A - I)\mathbf{v}_3 = \mathbf{v}_2$. To solve this, we row reduce the augmented system $[A - I \mid \mathbf{v}_2]$:

$$\left[\begin{array}{ccc|c} -2 & -3 & 2 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then we may take $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, again up to a multiple of $\mathbf{v}_1$. Hence, the final solution is of the form

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^t \left(t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) + c_3 e^t \left(\frac{t^2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right).$$

(d) $(1 + x)y\,dx + x\,dy = 0$

Solution: As in part (b), we check this DE for exactness: $M(x, y) = (1 + x)y$ and $N(x, y) = x$, so $M_y = 1 + x$ while $N_x = 1$. Since these are not equal, the DE is not exact. We instead isolate $y' = \frac{dy}{dx}$ as

$$y' = -\frac{(1 + x)y}{x} = -\left(1 + \frac{1}{x}\right) y,$$

so the DE is therefore separable. Separating and integrating, we have

$$\int \frac{1}{y}\,dy = \int 1 + \frac{1}{x}\,dx, \quad \Rightarrow \quad \ln|y| = -x - \ln|x| + C.$$

Then $y = Ce^{-x - \ln|x|} = \frac{Ce^{-x}}{x}$. We note that $y = 0$ is a solution that we excluded after the separation, but it is reincorporated into this general solution with $C = 0$.

(e) $x' = -x + 2y + t$, $y' = -x - 4y + 1 + t$

Solution: We see that this is a nonhomogeneous linear system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$, with $A = \begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} t \\ 1+t \end{bmatrix}$. We first find the complementary solution from the homogeneous equation, starting with the eigenvalues of A:

$$\det(A - \lambda I) = \begin{bmatrix} -1-\lambda & 2 \\ -1 & -4-\lambda \end{bmatrix} = \lambda^2 + 5\lambda + 6 = (\lambda+2)(\lambda+3).$$

Then the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -3$. We find eigenvectors for them: for $\lambda_1 = -2$, we row reduce $A + 2I$:

$$A + 2I = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

so $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Similarly, row reducing $A + 3I$,

$$A + 3I = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

so $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence, the complementary solution is

$$\mathbf{x}_c(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We now find a particular solution $\mathbf{x}_p(t)$ to match $\mathbf{f}(t)$. Rearranging the DE, it must therefore satisfy $\mathbf{x}' - A\mathbf{x} = \mathbf{f}(t)$. Since $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we take $\mathbf{x}(t) = \mathbf{a} + t\mathbf{b}$ for unknown constant vectors $\mathbf{a}$ and $\mathbf{b}$. Then $\mathbf{x}' = \mathbf{b}$, so the DE becomes $\mathbf{b} - A\mathbf{a} - A\mathbf{b}t = \mathbf{f}(t)$. Separating the t terms from the constants, we have the equations

$$-A\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{b} - A\mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The first equation involves only $\mathbf{b}$, so we solve it:

$$A^{-1} = \frac{1}{6} \begin{bmatrix} -4 & -2 \\ 1 & -1 \end{bmatrix}, \qquad \mathbf{b} = -A^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Next, $A\mathbf{a} = \mathbf{b} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so

$$\mathbf{a} = A^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -4 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

Hence, $\mathbf{x}_p = \begin{bmatrix} t - \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$, so the general solution to the nonhomogeneous equation is

$$\mathbf{x}_c(t) = \begin{bmatrix} t - \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} + c_1 e^{-2t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(f) $y' = (1-y)\cos x$, $y(\pi) = 2$

Solution: Distributing the $\cos x$ term on the right-hand side, we see that this equation is linear: $y' + y\cos x = \cos x$. Multiplying through by the integrating factor $\mu(x) = e^{\int \cos x \, dx} = e^{\sin x}$, this equation is

$$(e^{\sin x} y)' = (\cos x) e^{\sin x} \quad \Rightarrow \quad e^{\sin x} y = \int (\cos x) e^{\sin x} \, dx = e^{\sin x} + C.$$

Then $y = 1 + Ce^{-\sin x}$. Applying the initial condition $y(\pi) = 2$, $2 = 1 + Ce^0 = C + 1$, so $C = 1$, and $y = 1 + e^{-\sin x}$.

Alternately, the equation is also separable, as $\frac{1}{1-y} y' = \cos x$. Integrating, $-\ln|1-y| = \sin x + C$, so, isolating y, we have $y = 1 + Ce^{-\sin x}$.

2. Brine circulates at a rate of $r = 10$ gallons per minute between 3 tanks, from tank 1 to tank 2 to tank 3 and back to tank 1. The tanks have volumes $V_1 = 20$, $V_2 = 50$, and $V_3 = 20$ gallons, and at time t_0 tank 1 contains $x_0 = 18$ pounds of salt, while the other tanks contain none.

(a) Write a system of differential equations that governs the amounts of salt $x_i(t)$ in each tank.

Solution: The amount of salt per unit time transferred from one tank to another is the rate r times the concentration x_i/V_i, so adding up these quantities with the appropriate signs will give us the net rates of change of the x_i. Then

$$\begin{aligned} x_1' &= -\frac{r}{V_1} x_1 + \frac{r}{V_3} x_3 = -\frac{1}{2} x_1 + \frac{1}{2} x_3, \\ x_2' &= \frac{r}{V_1} x_1 - \frac{r}{V_2} x_2 = \frac{1}{2} x_1 - \frac{1}{5} x_2, \\ x_3' &= \frac{r}{V_2} x_2 - \frac{r}{V_3} x_3 = \frac{1}{5} x_2 - \frac{1}{2} x_3. \end{aligned}$$

(b) Find the general solution to this system of equations.

Solution: We write this system in the form $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix}$$

We compute the eigenvalues of A:

$$\det(A-\lambda I) = \begin{bmatrix} -\frac{1}{2}-\lambda & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5}-\lambda & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2}-\lambda \end{bmatrix}$$
$$= -(\frac{1}{2}+\lambda)^2(\frac{1}{5}+\lambda) + (\frac{1}{2})^2\frac{1}{5} = -\frac{1}{20}\lambda(20\lambda^2+24\lambda+9).$$

Finding the roots of this quadratic, we have $\lambda = \frac{-24\pm\sqrt{24^2-4(20)(9)}}{40} = -\frac{3}{5} \pm \frac{3}{10}i$, in addition to the $\lambda = 0$ from the λ factor. We find eigenvectors for $\lambda_1 = 0$ and $\lambda_2 = -\frac{3}{5} - \frac{3}{10}i$. Row reducing $A - 0I = A$,

$$A = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 5 & -2 & 0 \\ 0 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

Then letting $\mathbf{v} = \begin{bmatrix} a & b & c \end{bmatrix}^T$, $a = c$ and $2b = 5c$, so let $c = 2$; then $\mathbf{v} = \begin{bmatrix} 2 & 5 & 2 \end{bmatrix}^T$.

Row reducing $A + (\frac{3}{5} + \frac{3}{10}i)I$,

$$A - \lambda_2 I = \begin{bmatrix} \frac{1}{10}+\frac{3}{10}i & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{5}+\frac{3}{10}i & 0 \\ 0 & \frac{1}{5} & \frac{1}{10}+\frac{3}{10}i \end{bmatrix} \sim \begin{bmatrix} 1+3i & 0 & 5 \\ 5 & 4+3i & 0 \\ 0 & 2 & 1+3i \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & 0 & 1-3i \\ 10 & 8+6i & 0 \\ 0 & 2 & 1+3i \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1-3i \\ 10 & 0 & 5-15i \\ 0 & 2 & 1+3i \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1-3i \\ 0 & 2 & 1+3i \\ 0 & 0 & 0 \end{bmatrix}$$

Then $2a = (-1+3i)c$ and $2b = (-1-3i)c$, so letting $c = -2$, $\mathbf{v}_2 = \begin{bmatrix} 1-3i \\ 1+3i \\ -2 \end{bmatrix}$. We then obtain 2 lineearly independent solutions from the real and imaginary parts of $e^{\lambda_2 t}\mathbf{v}_2$:

$$e^{\lambda t}\mathbf{v}_2 = e^{-3/5t}(\cos\frac{3}{10}t - i\sin\frac{3}{10}t) \begin{bmatrix} 1-3i \\ 1+3i \\ -2 \end{bmatrix}$$
$$= e^{-3/5t} \begin{bmatrix} \cos\frac{3}{10}t - 3\sin\frac{3}{10}t \\ \cos\frac{3}{10}t + 3\sin\frac{3}{10}t \\ -2\cos\frac{3}{10}t \end{bmatrix} + ie^{-3/5t} \begin{bmatrix} -\sin\frac{3}{10}t - 3\cos\frac{3}{10}t \\ -\sin\frac{3}{10}t + 3\cos\frac{3}{10}t \\ 2\sin\frac{3}{10}t \end{bmatrix}.$$

Therefore, the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + c_2 e^{-3/5t} \begin{bmatrix} \cos \frac{3}{10}t - 3\sin \frac{3}{10}t \\ \cos \frac{3}{10}t + 3\sin \frac{3}{10}t \\ -2\cos \frac{3}{10}t \end{bmatrix} + c_3 e^{-3/5t} \begin{bmatrix} -\sin \frac{3}{10}t - 3\cos \frac{3}{10}t \\ -\sin \frac{3}{10}t + 3\cos \frac{3}{10}t \\ 2\sin \frac{3}{10}t \end{bmatrix}.$$

Note, of course, that the two trigonometric terms may appear in different, equivalent linear combinations, depending on the choice of eigenvalue in the complex pair and eigenvector for that eigenvalue.

(c) Find the solution matching the initial condition stated above.

Solution: Finally, we match the initial condition $\mathbf{x}(0) = \begin{bmatrix} 18 & 0 & 0 \end{bmatrix}^T$: evaluating the general solution at $t = 0$ gives the linear system

$$c_1 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}$$

We then row reduce the corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 18 \\ 5 & 1 & 3 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 3 & -3 & 18 \\ 0 & 6 & 3 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 3 & -12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

Therefore, the solution matching this initial condition is

$$\mathbf{x}(t) = 2 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + 2e^{-3/5t} \begin{bmatrix} \cos \frac{3}{10}t - 3\sin \frac{3}{10}t \\ \cos \frac{3}{10}t + 3\sin \frac{3}{10}t \\ -2\cos \frac{3}{10}t \end{bmatrix} - 4e^{-3/5t} \begin{bmatrix} -\sin \frac{3}{10}t - 3\cos \frac{3}{10}t \\ -\sin \frac{3}{10}t + 3\cos \frac{3}{10}t \\ 2\sin \frac{3}{10}t \end{bmatrix}$$
$$= \begin{bmatrix} 4 + e^{-3/5t}(14\cos \frac{3}{10}t - 2\sin \frac{3}{10}t) \\ 10 + e^{-3/5t}(-10\cos \frac{3}{10}t + 10\sin \frac{3}{10}t) \\ 4 + e^{-3/5t}(-4\cos \frac{3}{10}t - 8\sin \frac{3}{10}t) \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 7 & 4 \\ -1 & 3 \end{bmatrix}$.

(a) Find 2 linearly independent solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ to the system $\mathbf{x}' = A\mathbf{x}$ using eigenvalue and eigenvector techniques. Show that the solutions you determine are actually linearly independent.

Solution: We compute the eigenvalues of A:

$$\det(A - \lambda I) = \begin{bmatrix} 7-\lambda & 4 \\ -1 & 3-\lambda \end{bmatrix} = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2.$$

Then $\lambda = 5$ is the only eigenvalue, of multiplicity 2. We compute its eigenvectors, by row reducing $A - 5I$:

$$A - 5I = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Then $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is the only eigenvector for this eigenvalue, up to scale. We therefore look for a generalized eigenvector $\mathbf{v}_2$ such that $(A - 5I)\mathbf{v}_2 = \mathbf{v}_1$, which we find by row reduction of $[A - 5I \mid \mathbf{v}_1]$:

$$\left[\begin{array}{cc|c} 2 & 4 & 2 \\ -1 & -2 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Then one choice for $\mathbf{v}_2$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. With this complete set of generalized eigenvectors, we have solutions

$$\mathbf{x}_1(t) = e^{5t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \mathbf{x}_2(t) = e^{5t} \begin{bmatrix} 2t+1 \\ -t \end{bmatrix}$$

We also check that these solutions are linearly independent by computing the Wronskian:

$$W(t) = \begin{vmatrix} 2e^{5t} & (2t+1)e^{5t} \\ -e^{5t} & -te^{5t} \end{vmatrix} = -2te^{10t} + (2t+1)e^{10t} = e^{10t}$$

Since $W(t) \neq 0$ for all t, these solutions are indeed linearly independent.

(b) Form a fundamental matrix $\Phi(t)$ from these two solutions and use it to compute the matrix exponential e^{At}.

Solution: Define the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{5t} & (2t+1)e^{5t} \\ -e^{5t} & -te^{5t} \end{bmatrix}.$$

Then

$$\Phi(0) = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \qquad \Phi(0)^{-1} = \frac{1}{1} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix},$$

so therefore

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} 2e^{5t} & (2t+1)e^{5t} \\ -e^{5t} & -te^{5t} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (2t+1)e^{5t} & 4te^{5t} \\ -te^{5t} & (1-2t)e^{5t} \end{bmatrix}$$

(c) Use that $A = 5I + B$, where B is a nilpotent matrix, to compute e^{At} directly, without requiring the eigenvalues and eigenvectors computed above. Check that you obtain the same answer as in the previous part.

Solution: We check that $B = A - 5I = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$ is nilpotent:

$$B^2 = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then $e^{Bt} = I + Bt$, so because $5I$ and B commute, $e^{At} = e^{5It}e^{Bt} = e^{5t}(I + Bt)$. Expanding this,

$$e^{At} = e^{5t} \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix},$$

which is the same as the result computed in part (b).

(d) Use variation of parameters to find the unique solution to the nonhomogeneous initial value problem

$$\mathbf{x}' = \begin{bmatrix} 7 & 4 \\ -1 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ e^{5t} \end{bmatrix}, \qquad \mathbf{x}(2) = \begin{bmatrix} 7e^{10} \\ -2e^{10} \end{bmatrix}$$

Solution: We first find a particular solution $\mathbf{x}_p(t)$ to this nonhomogeneous system via variation of parameters. Then

$$\begin{aligned} \mathbf{x}_p(t) &= e^{At} \int e^{-At}\mathbf{f}(t)\,dt = e^{At} \int e^{-5t} \begin{bmatrix} 1-2t & -4t \\ t & 1+2t \end{bmatrix} \begin{bmatrix} 0 \\ e^{5t} \end{bmatrix} dt = e^{At} \int \begin{bmatrix} -4t \\ 1+2t \end{bmatrix} dt \\ &= e^{At} \begin{bmatrix} -2t^2 \\ t+t^2 \end{bmatrix} = e^{5t} \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix} \begin{bmatrix} -2t^2 \\ t+t^2 \end{bmatrix} \\ &= e^{5t} \begin{bmatrix} -2t^2(1+2t) + 4t(t+t^2) \\ 2t^3 + (t+t^2)(1-2t) \end{bmatrix} = e^{5t} \begin{bmatrix} 2t^2 \\ t-t^2 \end{bmatrix} \end{aligned}$$

The general solution is then $\mathbf{x}(t) = e^{At}\mathbf{c} + \mathbf{x}_p(t)$. At $t = 2$, $e^{2A}\mathbf{c} = \mathbf{x}(2) - \mathbf{x}_p(2)$, so $\mathbf{c} = e^{-2A}(\mathbf{x}(2) - \mathbf{x}_p(2))$. Since $\mathbf{x}(2) - \mathbf{x}_p(2) = \begin{bmatrix} 7e^{10} \\ -2e^{10} \end{bmatrix} - \begin{bmatrix} 8e^{10} \\ -2e^{10} \end{bmatrix} = \begin{bmatrix} -e^{10} \\ 0 \end{bmatrix}$,

$$\mathbf{c} = e^{-2A}(\mathbf{x}(2) - \mathbf{x}_p(2)) = e^{-10} \begin{bmatrix} -3 & -8 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -e^{10} \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Then the solution matching the initial condition is

$$\mathbf{x}(t) = e^{5t} \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} + e^{5t} \begin{bmatrix} 2t^2 \\ t-t^2 \end{bmatrix} = e^{5t} \begin{bmatrix} 3-2t+2t^2 \\ -2+2t-t^2 \end{bmatrix}.$$

4. Consider the higher-order linear system $x'' - 5x' + 9x - 3y = 0$, $y' + 2y - 5x = 0$.

(a) Introduce the variable $z = x'$ and rewrite this system as a first-order system.

Solution: Let $z = x'$. Then the first-order system is $x' = z$, $y' = 5x - 2y$, $z' = -9x + 3y + 5z$.

(b) Find the general solution $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ of the system.

Solution: We write the system as $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} 0 & 0 & 1 \\ 5 & -2 & 0 \\ -9 & 3 & 5 \end{bmatrix}$. We then compute the eigenvalues of A:

$$\det(A - \lambda I) = \begin{bmatrix} -\lambda & 0 & 1 \\ 5 & -2-\lambda & 0 \\ -9 & 3 & 5-\lambda \end{bmatrix} = -\lambda \begin{bmatrix} -2-\lambda & 0 \\ 3 & 5-\lambda \end{bmatrix} + \begin{bmatrix} 5 & -2-\lambda \\ -9 & 3 \end{bmatrix}$$
$$= -\lambda(-2-\lambda)(5-\lambda) + 15 - 9(2+\lambda) = -\lambda^3 + 3\lambda^2 + \lambda - 3$$
$$= -(\lambda - 3)(\lambda - 1)(\lambda + 1)$$

Then A has distinct real eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$, and $\lambda_3 = -1$, for which we find eigenvectors $\mathbf{v} = \begin{bmatrix} a & b & c \end{bmatrix}^T$. First, row reducing $A - \lambda I$ with $\lambda_1 = 3$,

$$A - 3I = \begin{bmatrix} -3 & 0 & 1 \\ 5 & -5 & 0 \\ -9 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & 1 \\ 0 & -6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $a = b$ and $3b = c$, so taking $c = 3$, $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 & 3 \end{bmatrix}^T$. Next, for $\lambda_2 = 1$,

$$A - I = \begin{bmatrix} -1 & 0 & 1 \\ 5 & -3 & 0 \\ -9 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 5 \\ 0 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

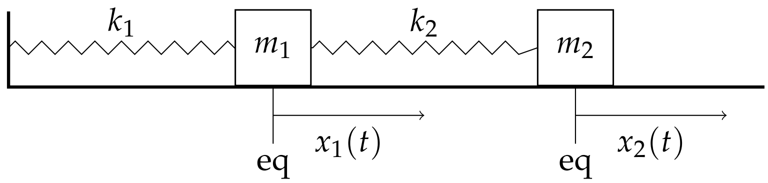
Then $a = c$ and $3b = 5c$, so taking $c = 3$, $\mathbf{v}_2 = \begin{bmatrix} 3 & 5 & 3 \end{bmatrix}^T$. Finally, for $\lambda_3 = -1$,

$$A + I = \begin{bmatrix} 1 & 0 & 1 \\ 5 & -1 & 0 \\ -9 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -5 \\ 0 & 3 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $a = -c$ and $b = -5c$, so taking $c = -1$, $\mathbf{v}_2 = \begin{bmatrix} 1 & 5 & -1 \end{bmatrix}^T$. Thus, the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

5. Consider the coupled mass-spring system pictured below:



Assume that $m_1 = 2$ kg, $m_2 = 1$ kg, $k_1 = 16$ N/m, $k_2 = 8$ N/m.

(a) Use force diagrams and Newton's second law ($F = ma$) on each mass to write a set of coupled second-order linear DEs describing the displacements $x_1(t)$ and $x_2(t)$.

Solution: The forces acting on mass m_1 are k_1x_1 to the left and $k_2(x_2 - x_1)$ to the right, so

$$m_1x_1'' = -k_1x_1 + k_2(x_2 - x_1) = -(k_1 + k_2)x_1 + k_2x_2.$$

The only force acting on mass m_2 is $k_2(x_2 - x_1)$ to the left, so

$$m_2x_2'' = -k_2(x_2 - x_1) = k_2x_1 + k_2x_2.$$

Plugging in the values for the masses and spring constants and normalizing,

$$x_1'' = -12x_1 + 4x_2, \qquad x_2'' = 8x_1 - 8x_2.$$

(b) Using new variables $y_1 = x_1'$ and $y_2 = x_2'$, rewrite the system as a homogeneous first-order system in 4 variables. Find the eigenvalues and eigenvectors of the resulting system and use them to write a general solution to the system.

Solution: Adding these variables and writing the equation as $\mathbf{x}' = A\mathbf{x}$,

$$\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_1 \\ y_2 \\ -12x_1 + 4x_2 \\ 8x_1 - 8x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12 & 4 & 0 & 0 \\ 8 & -8 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}.$$

We find the eigenvalues of this matrix A:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -12 & 4 & -\lambda & 0 \\ 8 & -8 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 & 1 \\ 4 & -\lambda & 0 \\ -8 & 0 & -\lambda \end{vmatrix} + \begin{vmatrix} 0 & -\lambda & 1 \\ -12 & 4 & 0 \\ 8 & -8 & -\lambda \end{vmatrix}$$

$$= -\lambda((-\lambda)^3 + 1(\lambda)(-8))) + 12\lambda^2 + ((-12)(-8) - (4)(8))$$

$$= \lambda^4 + 20\lambda^2 + 64 = (\lambda^2 + 16)(\lambda^2 + 4).$$

Then the eigenvalues are $\lambda = \pm 2i$ and $\lambda = \pm 4i$. We compute eigenvectors $\mathbf{v} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$ for $\lambda = -2i$ and $\lambda = -4i$. Row reducing $A + 2iI$,

$$A + 2iI = \begin{bmatrix} 2i & 0 & 1 & 0 \\ 0 & 2i & 0 & 1 \\ -12 & 4 & 2i & 0 \\ 8 & -8 & 0 & 2i \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -i & 0 \\ 0 & 2 & 0 & -i \\ 0 & 4 & -4i & 0 \\ 0 & -8 & 4i & 2i \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & 0 & -i & 0 \\ 0 & 2 & 0 & -i \\ 0 & 0 & -4i & 2i \\ 0 & 0 & 4i & -2i \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -i & 0 \\ 0 & 2 & 0 & -i \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 0 & -i \\ 0 & 2 & 0 & -i \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $4a = id$, $2b = id$, and $2c = d$, so taking $d = 4$, $\mathbf{v}_1 = \begin{bmatrix} i & 2i & 2 & 4 \end{bmatrix}^T$. For $\lambda = -4i$,

$$A + 4iI = \begin{bmatrix} 4i & 0 & 1 & 0 \\ 0 & 4i & 0 & 1 \\ -12 & 4 & 4i & 0 \\ 8 & -8 & 0 & 4i \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & -i & 0 \\ 0 & 4 & 0 & -i \\ 0 & 4 & i & 0 \\ 0 & -8 & -2i & 4i \end{bmatrix}$$
$$\sim \begin{bmatrix} 4 & 0 & -i & 0 \\ 0 & 4 & 0 & -i \\ 0 & 0 & i & i \\ 0 & 0 & -2i & -2i \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 0 & i \\ 0 & 4 & 0 & -i \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $4a = -id$, $4b = id$, and $c = -d$, so taking $d = -4$, $\mathbf{v}_2 = \begin{bmatrix} i & -i & 4 & -4 \end{bmatrix}^T$. We get a pair of solutions from each eigenvector: first,

$$e^{-2it}\mathbf{v}_1 = (\cos 2t - i\sin 2t)\begin{bmatrix} i \\ 2i \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \sin 2t \\ 2\sin 2t \\ 2\cos 2t \\ 4\cos 2t \end{bmatrix} + i\begin{bmatrix} \cos 2t \\ 2\cos 2t \\ -2\sin 2t \\ -4\sin 2t \end{bmatrix}$$
$$e^{-4it}\mathbf{v}_2 = (\cos 4t - i\sin 4t)\begin{bmatrix} i \\ -i \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} \sin 4t \\ -\sin 4t \\ 4\cos 4t \\ -4\cos 4t \end{bmatrix} + i\begin{bmatrix} \cos 4t \\ -\cos 4t \\ -4\sin 4t \\ 4\sin 4t \end{bmatrix}$$

Hence, the general solution is

$$\mathbf{x}(t) = c_1\begin{bmatrix} \sin 2t \\ 2\sin 2t \\ 2\cos 2t \\ 4\cos 2t \end{bmatrix} + c_2\begin{bmatrix} \cos 2t \\ 2\cos 2t \\ -2\sin 2t \\ -4\sin 2t \end{bmatrix} + c_3\begin{bmatrix} \sin 4t \\ -\sin 4t \\ 4\cos 4t \\ -4\cos 4t \end{bmatrix} + c_4\begin{bmatrix} \cos 4t \\ -\cos 4t \\ -4\sin 4t \\ 4\sin 4t \end{bmatrix}.$$

(c) Find frequencies ω_1, ω_2 and constant vectors $\mathbf{v}_1, \mathbf{v}_2$ so that the general solution to $x_1(t)$ and $x_2(t)$ is in the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \cos(\omega_1 t - \alpha_1)\mathbf{v}_1 + c_2 \cos(\omega_2 t - \alpha_2)\mathbf{v}_2,$$

where the amplitudes c_1, c_2 and the phases α_1, α_2 are parameters depending on the initial values of x_1, x_2, and their derivatives.

Solution: We isolate the x_1 and x_2 components:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \sin 2t + c_2 \cos 2t + c_3 \sin 4t + c_4 \cos 4t \\ 2c_1 \sin 2t + 2c_2 \cos 2t - c_3 \sin 4t - c_4 \cos 4t \end{bmatrix}$$
$$= C_1 \cos(2t - \alpha_1) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \cos(4t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus, the frequencies are 2 and 4, and the vectors are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

6. Each of the following linear or affine-linear systems has a single critical point. Find this critical point and characterize its type and stability.

(a) $x' = 2x - y, y' = 3x - 2y$

Solution: Since there are no constant terms in this system, the only critical point is at $(0,0)$. Writing the system as $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$. We compute the eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1).$$

Then A has the distinct real eigenvalues -1 and 1. Since one is positive and the other negative, the critical point is a saddle point, which is unstable.

(b) $x' = 2x - 5y - 1, y' = x - 2y - 1$

Solution: We first find the critical point of this system. Let $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$. Then we must solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which we do by row reduction:

$$\left[\begin{array}{cc|c} 2 & -5 & 1 \\ 1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

Therefore, the critical point is at $(3,1)$. The linearization of the system at this critical point is just A, so we find its eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 1$$

The characteristic polynomial has roots $\lambda = \pm i$, so the eigenvalues are a pure imaginary pair, and the system has a (stable) center at $(3,1)$.

(c) $x' = -x + 4y, y' = -2x + 3y$
Solution: Since there are no constant terms in this system, the only critical point is at $(0, 0)$. Writing the system as $\mathbf{x}' = A\mathbf{x}$, $A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}$. We compute the eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 4 \\ -2 & 3-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5$$

Using the quadratic formula or inspection, we see that the eigenvalues are then $\lambda = 1 \pm 2i$. Since this is a complex pair with positive real part, the critical point is an unstable spiral point.

(d) $x' = x - 2y, y' = 3x - 4y - 2$
Solution: We first find the critical point of this system. Let $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$. Then we must solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, which we do by row reduction:

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 3 & -4 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

Therefore, the critical point is at $(2, 1)$. The linearization of the system at this critical point is just A, so we find its eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

Then A has the distinct real eigenvalues -1 and -2. Since both values are negative, the critical point is an asymptotically stable node.

7. Find all of the critical points of the autonomous system $x' = (y - x)(1 - x - y)$, $y' = x(2 + y)$. Linearize the system at each critical point to characterize its type and stability, if possible.
Solution: We first find the critical points of the system. From the x' equation, we see that either $y = x$ or $y = 1 - x$, and from the y' equation, we see that either $x = 0$ or $y = -2$. We address these different cases:

- If $x = 0$ and $y = x$, then $y = 0$, so $(0, 0)$ is a critical point.
- If $x = 0$ and $y = 1 - x$, then $y = 1$, so $(0, 1)$ is a critical point.
- If $y = -2$ and $y = x$, then $x = -2$, so $(-2, -2)$ is a critical point.
- If $y = -2$ and $y = 1 - x$, then $x = 3$, so $(3, -2)$ is a critical point.

Then $(0, 0)$, $(0, 1)$, $(-2, -2)$, and $(3, -2)$ are the four critical points of this system. We characterize the system at each of them. First, we construct the Jacobian matrix of the system:

$$x' = F(x,y) = (y-x)(1-x-y) = y - y^2 - x + x^2$$
$$y' = G(x,y) = 2x + xy,$$
$$J(x,y) = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 2x-1 & 1-2y \\ 2+y & x \end{bmatrix}$$

We then compute the eigenvalues of $J(x,y)$ at each critical point:

- At $(0,0)$,
$$\det(J(0,0) - \lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1)$$
Since the eigenvalues are -2 and 1, this critical point is a saddle point, which is unstable.
- At $(0,1)$,
$$\det(J(0,1) - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 3 & -\lambda \end{vmatrix} = \lambda^2 + \lambda + 3.$$
The roots of this characteristic polynomial are $\lambda = \frac{-1\pm\sqrt{1-12}}{2} = \frac{-1\pm i\sqrt{11}}{2}$. Since these are complex with negative real part, the system has an asymptotically stable spiral point here.
- At $(-2,-2)$, $J(-2,-2) = \begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix}$, which is upper triangular. Therefore, the eigenvalues are the values on the main diagonal, -5 and -2. Since these real values are both negative, the system has an asymptotically stable node at this point.
- At $(3,-2)$, $J(3,-2) = \begin{bmatrix} 5 & 5 \\ 0 & 3 \end{bmatrix}$, which is upper triangular. Therefore, the eigenvalues are the values on the main diagonal, 5 and 3. Since these real values are both positive, the system has an unstable node at this point.

8. The linear system $\mathbf{x}' = A\mathbf{x}$ has the general solution
$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Use this information to recover A. (Hint: $\Phi'(t) = A\Phi(t)$.)

Solution: Since any fundamental matrix $\Phi(t)$ satisfies the matrix differential equation $\Phi'(t) = A\Phi(t)$, we may evaluate this at $t = 0$ to obtain $\Phi'(0) = A\Phi(0)$. Isolating A, $A = \Phi'(0)\Phi(0)^{-1}$. From the general solution, we obtain one choice of $\Phi(t)$:

$$\Phi(t) = \begin{bmatrix} 4e^{2t} & 3e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix} \qquad \Phi'(t) = \begin{bmatrix} 8e^{2t} & -3e^{-t} \\ 2e^{2t} & -e^{-t} \end{bmatrix}$$
$$\Phi(0) = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \qquad \Phi'(0) = \begin{bmatrix} 8 & -3 \\ 2 & -1 \end{bmatrix}$$

Using the formula for the inverse of a 2×2 matrix, we find that

$$\Phi(0)^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}.$$

Then

$$A = \Phi'(0)\Phi(0)^{-1} = \begin{bmatrix} 8 & -3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 11 & -36 \\ 3 & -10 \end{bmatrix}.$$