Midterm #1 — March 6, 2013, 10:00 to 10:53 AM

Name: Solution Key

Circle your recitation:

R01 (Claudio · Fri) R02 (Xuan · Wed) R03 (Claudio · Mon)

- **You have a maximum of 53 minutes.** This is a closed-book, closed-notes exam. No calculators or other electronic aids are allowed.
- Read each question carefully. Show your work and justify your answers for full credit. You do not need to simplify your answers unless instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

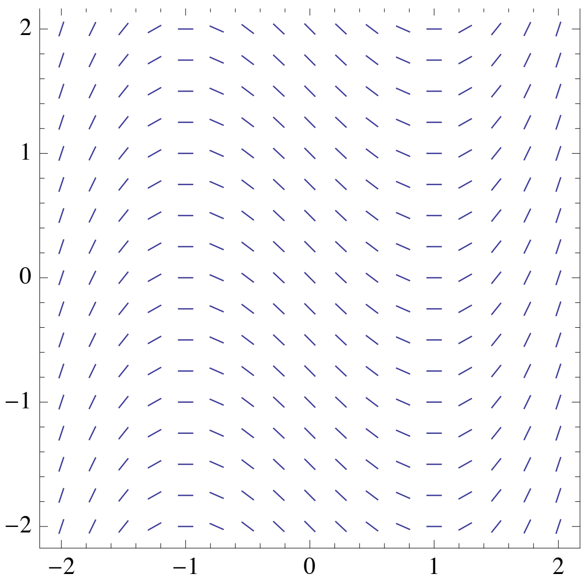
Grading

1	/10
2	/15
3	/15
4	/15
5	/15
6	/15
7	/15
Total	/100

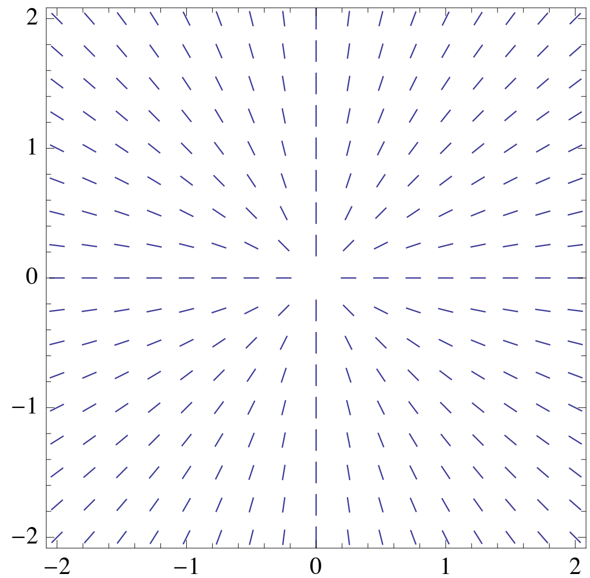
1. (*10 points*)
For each of the slope fields below, identify the differential equation A–E matching it:

A. $y' = y^2 - 1$ B. $xy' = y$ C. $y' = xy$ D. $y' = x^2 - 1$ E. $y' = y - x$

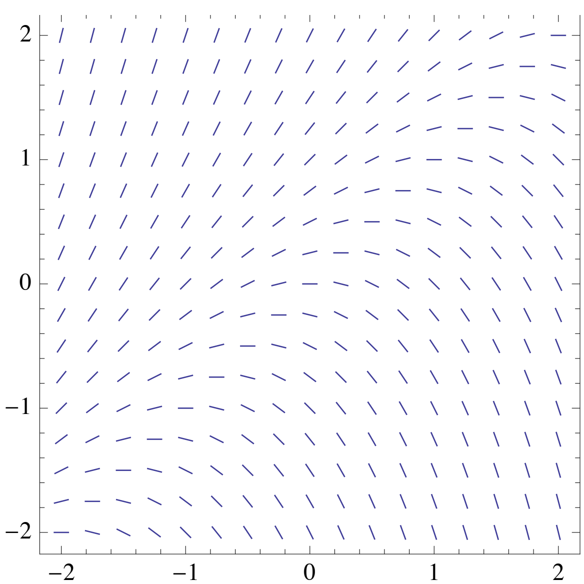
I. D



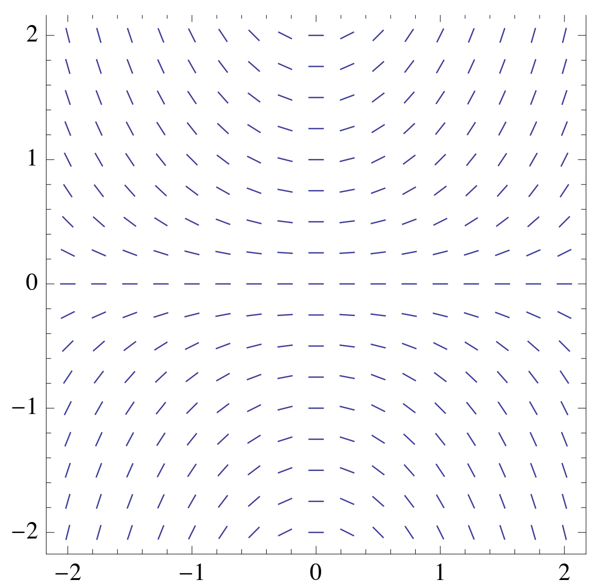
II. B



III. E



IV. C



2. (*15 points*) Solve the initial value problem $xy' = 4y + 4x^6$, $y(1) = 1$.

Solution: This is a linear equation, which normalizes to

$$y' - \frac{4}{x}y = 4x^5.$$

The integrating factor is

$$\mu(x) = e^{\int -\frac{4}{x}\,dx} = e^{-4\ln|x|} = x^{-4}.$$

Multiplying by this integrating factor, the equation becomes

$$(x^{-4}y)' = x^{-4}y' - 4x^{-5}y = 4x,$$

so integrating gives $x^{-4}y = 2x^2 + C$, and $y = 2x^6 + Cx^4$. Applying the initial condition,

$$1 = 2(1)^6 + C(1)^4 = 2 + C,$$

so $C = -1$, and $y = 2x^6 - x^4$.

3. (*15 points*) Find the general solution to the differential equation $2xyy' + y^2 + x^2 = 0$.

Solution: We can normalize the DE to $y' + \frac{y}{2x} + \frac{x}{2y} = 0$, which is homogeneous. Making the substitution $v = y/x$, $y = xv$, and $y' = xv' + v$, we have the separable DE

$$xv' + v + \frac{v}{2} + \frac{2}{v} \quad \Rightarrow \quad xv' = -\frac{1+3v^2}{2v} \quad \Rightarrow \quad \frac{2v}{1+3v^2}v' = -\frac{1}{x}.$$

Integrating with respect to x, and recognizing the logarithmic derivative in v,

$$\frac{1}{3}\ln|1+3v^2| = C - \ln|x| \quad \Rightarrow \quad \ln|1+3v^2| + 3\ln|x| = C \quad \Rightarrow \quad x^3(1+3v^2) = C,$$

redefining C throughout. Recalling $v = y/x$, we have $x^3 + 3xy^2 = C$. Since y appears only once, we can solve for it explicitly. Dividing by 3 and rescaling C,

$$y^2 = \frac{C}{x} - \frac{x^2}{3} \quad \Rightarrow \quad y = \pm\sqrt{\frac{C}{3x} - \frac{x^2}{3}}.$$

Alternately, we note that the normalized DE is $y' + \frac{1}{2x}y = -\frac{x}{2}y^{-1}$, which is Bernoulli with $n = 1$. Let $u = y^{1-n} = y^2$, so $y = u^{1/2}$, and $y' = \frac{1}{2}u^{-1/2}u'$. Then

$$\frac{1}{2}u^{-1/2}u' + \frac{1}{2x}u^{1/2} = -\frac{x}{2}u^{-1/2} \quad \Rightarrow \quad u' + \frac{1}{x}u = -x,$$

which is linear. Using the integration factor $\mu(x) = e^{\int \frac{1}{x}\,dx} = x$, $(xu)' = xu' + u = -x^2$, so integrating gives $xu = C - x^3/3$. Since $u = y^2$,

$$y^2 = \frac{C}{x} - \frac{x^2}{3}. \quad \Rightarrow \quad y = \pm\sqrt{\frac{C}{3x} - \frac{x^2}{3}}.$$

Finally, writing the equation in differential form, it becomes $(y^2 + x^2)\,dx + 2xy\,dy = 0$. Letting $M(x,y) = y^2 + x^2$ and $N(x,y) = 2xy$, we compare $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$:

$$\frac{\partial M}{\partial y} = (y^2 + x^2)_y = 2y, \qquad \frac{\partial N}{\partial x} = (2xy)_x = 2y.$$

Since these are equal, the DE is exact. We then look for an $F(x,y)$ so that $F_x = M$ and $F_y = N$. Integrating M,

$$F(x,y) = \int y^2 + x^2\,dx = xy^2 + \frac{1}{3}x^3 + g(y),$$

where $g(y)$ is an undetermined function of y. Then $F_y = N$, so $2xy + g'(y) = 2xy$, and $g(y)$ is constant. Hence, we let $F(x,y) = xy^2 + \frac{1}{3}x^3$, and implicit solutions are given by $xy^2 + \frac{1}{3}x^3 = C$ for some C. Fortunately, we can solve for y explicitly:

$$y^2 = \frac{C}{x} - \frac{x^2}{3} \quad \Rightarrow \quad y = \pm\sqrt{\frac{C}{x} - \frac{x^2}{3}}.$$

4. (*15 points*) A cup of hot coffee at 190°F is placed outside, where it is currently 30°F. After 10 minutes, the coffee has cooled to 150°F.

(a) (*5 points*) Write a differential equation governing the temperature T of the coffee as a function of the time t since it is placed outside. Explain any parameters you include.

Solution: Assuming the coffee temperature obeys Newton's law of cooling, $\frac{dT}{dt} = -k(T - A)$, where k is a constant of proportionality, and $A = 30$ is the ambient temperature.

(b) (*5 points*) Find $T(t)$.

Solution: This DE is $T' + kT = 30k$, so using the integrating factor $\mu(t) = e^{kt}$, $e^{kt}T = \int 30ke^{kt}\,dt = 30e^{kt} + C$, and $T(t) = 30 + Ce^{-kt}$. At $t = 0$, $T(0) = 190 = 30 + C$, so $C = 160$. Then

$$T(t) = T(t) = 30 + 160e^{-kt}.$$

At $t = 10$, $T(10) = 150 = 30 + 160e^{-10k}$, so $e^{-10k} = \frac{150-30}{160} = \frac{120}{160} = \frac{3}{4}$, and $k = -\frac{1}{10}\ln\frac{3}{4}$. Then

$$T(t) = T(t) = 30 + 160e^{t/10 \ln 3/4} = 30 + 160\left(\frac{3}{4}\right)^{t/10}.$$

(c) (*5 points*) What is the temperature of the coffee after another 10 minutes?

Solution: At this time, $t = 20$, so the temperature is $T(20) = 30 + 160\left(\frac{3}{4}\right)^{20/10} = 30 + 160\frac{9}{16} = 30 + 90 = 120$°F.

5. (*15 points*) Find the solution $y(x)$ to the initial value problem $yy'' = (y')^2 - 6y'$ with $y(0) = 1$, $y'(0) = 8$.

Solution: We recognize this equation as a second-order reducible DE with no x. We then let $p(y) = y'$, so $y'' = p\frac{dp}{dy} = pp'$. Then the DE becomes

$$ypp' = p^2 - 6p \quad \Rightarrow \quad p' = \frac{1}{y}(p-6) \quad \Rightarrow \quad p' - \frac{1}{y}p = -\frac{6}{y},$$

which is linear (and separable, actually). Using the integrating factor $\mu(y) = e^{\int -\frac{1}{y}\,dy} = y^{-1}$, $(p/y)' = -6y^{-2}$, so

$$\frac{p}{y} = \frac{6}{y} + C \quad \Rightarrow \quad y' = p = Cy + 6.$$

Using the conditions that $y(0) = 1$ and $y'(0) = 8$, $8 = C - 6$, so $C = 2$. We now solve the linear (and, again, separable) DE $y' = 2y + 6$, or

$$y' - 2y = 6.$$

With the integrating factor $\rho(x) = e^{-2x}$, $(e^{-2x}y)' = 6e^{-2x}$, so

$$e^{-2x}y = -3e^{-2x} + D \quad \Rightarrow \quad y = De^{2x} - 3.$$

Since $y(0) = 1$, $1 = D(1) - 3$, so $D = 4$. Therefore, $y(x) = 4e^{2x} - 3$.

6. (*15 points*) Consider the differential equation $y' = 4y^2 - y^3 - 4y$.

(a) (*6 points*) Find the equilibria of this differential equation.

Solution: Since this DE $y' = f(y)$ is autonomous, we solve $f(y) = 0$: $4y^2 - y^3 - 4y = 0$, so $-y(y^2 - 4y + 4) = -y(y-2)^2 = 0$. Thus, $y = 0$ or $y = 2$, so these are the equilibria.

(b) (*3 points*) Construct a phase diagram for this differential equation.

Solution: Using that $f(1) = 4 - 1 - 4 = -1$, $f(3) = -3(3-2)^2 = -3$, and $f(-1) = -(-1)(-3)^2 = 9$, we have

$$\begin{array}{cccccl} + & 0 & - & 0 & - & y' \text{ sign} \\ \hline & \bullet & & \bullet & & \to y \\ \longrightarrow & 0 & \longleftarrow & 2 & \longleftarrow & \end{array}$$

(c) (*3 points*) Characterize the stability of the equilibria you found in part (a).

Solution: From the phase diagram, we see that $y = 0$ is stable, while $y = 2$ is semistable (and therefore is unstable.)

(d) (*3 points*) Describe the expected behavior of a solution $y(t)$ with initial value $y(0) = 4$ as t increases.

Solution: As t increases, $y'(t)$ is negative, so $y(t)$ decreases from 4 into the equilibrium $y = 2$. It approaches this value, but never actually reaches it, since solutions to this DE cannot cross by the existence and uniqueness theorem.

7. (*15 points*) Consider the following two differential equations:

$$\textbf{(I)} \quad (x+2y)\,dx + y\,dy = 0 \qquad\qquad \textbf{(II)} \quad (x+2y)\,dx + (2x-3y)\,dy = 0$$

(a) (*7 points*) Which of the equations is exact? Explain.
Solution: We check the derivatives of the coefficient functions in the DEs for exactness:

- **(I)** $\frac{\partial}{\partial y}(x+2y) = 2$, but $\frac{\partial}{\partial x}(y) = 0$, so **(I)** is not exact.
- **(II)** $\frac{\partial}{\partial y}(x+2y) = 2$ and $\frac{\partial}{\partial x}(2x-3y) = 2$, so **(II)** is exact.

(b) (*8 points*) Find the general solution to one of the equations.
Solution: We integrate the coefficient functions in **(II)** to solve it:

$$F(x,y) = \int x + 2y\,dx = \frac{1}{2}x^2 + 2xy + g(y), \qquad \frac{\partial F}{\partial y} = 2x + g'(y) = 2x - 3y.$$

Then $g(y) = -\frac{3}{2}y^2$, so $F(x,y) = \frac{1}{2}x^2 + 2xy - \frac{3}{2}y^2$, and implicit solutions are given by

$$\frac{1}{2}x^2 + 2xy - \frac{3}{2}y^2 = C,$$

or, rescaling by 2, $x^2 + 4xy - 3y^2 = C$.

(c) (*Extra credit: 5 points*) Find the general solution to the other equation.
Solution: Normalizing **(I)**, $y' + \frac{x}{y} + 2 = 0$, which is homogeneous. Using the substitution $v = y/x$, with $y' = xv' + v$, this becomes

$$xv' + v + \frac{1}{v} + 2 = xv' + \frac{v^2 + 2v + 1}{v} = 0$$

Then $xv' = -\frac{(v+1)^2}{v}$, so $\frac{v}{(v+1)^2}v' = -\frac{1}{x}$. We use partial fractions to rewrite $\frac{v}{(v+1)^2}$ as $\frac{1}{v+1} - \frac{1}{(v+1)^2}$. Integrating,

$$\ln|v+1| + \frac{1}{v+1} = -\ln|x| + C;$$

exponentiating, combining terms, and reintroducing y,

$$(y+x)e^{\frac{x}{y+x}} = C,$$

which cannot be solved for y explicitly.