Homework #2 Solutions

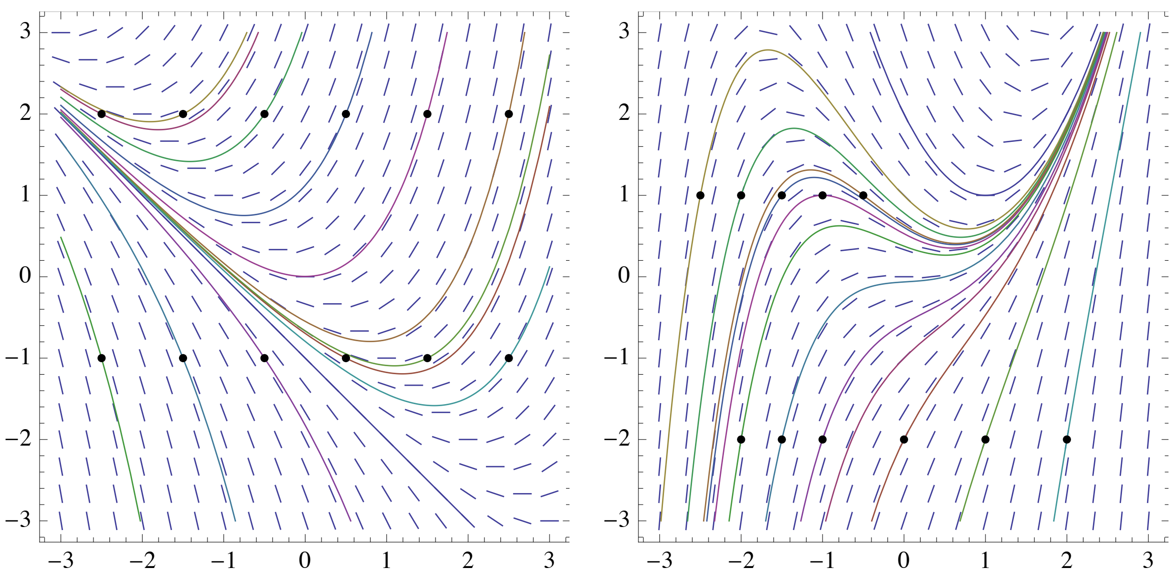
Problems

- Section 1.3: 2, 8, 12, 14, 28
- Section 1.5: 1, 2, 12, 14, 22, 36
- Extra Problem #1

1.3.2. Sketch likely solution curves through the given slope field for $\frac{dy}{dx} = x + y$.

1.3.8. Sketch likely solution curves through the given slope field for $\frac{dy}{dx} = x^2 - y$.

Solution: Here are slope fields with some solutions for these two problems:



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1.3.12. Determine whether Theorem 1.3.1 does or does not guarantee existence and/or uniqueness for the IVP $y' = x \ln y$, $y(1) = 1$. If a solution does exist, is it unique?

Solution: We see that the DE is already in normal form, with $f(x, y) = x \ln y$. We then examine this function and its derivative $f_y(x, y) = \frac{x}{y}$ around $x = 1$ and $y = 1$. Since both functions are defined and continuous in a region around $(1, 1)$, this IVP satisfies the hypotheses of Theorem 1, so it is guaranteed to have a unique solution $y(x)$ on some interval containing $x = 1$. ■

1.3.14. Determine whether Theorem 1.3.1 does or does not guarantee existence and/or uniqueness for the IVP $y' = \sqrt[3]{y}$, $y(0) = 0$. If a solution does exist, is it unique?

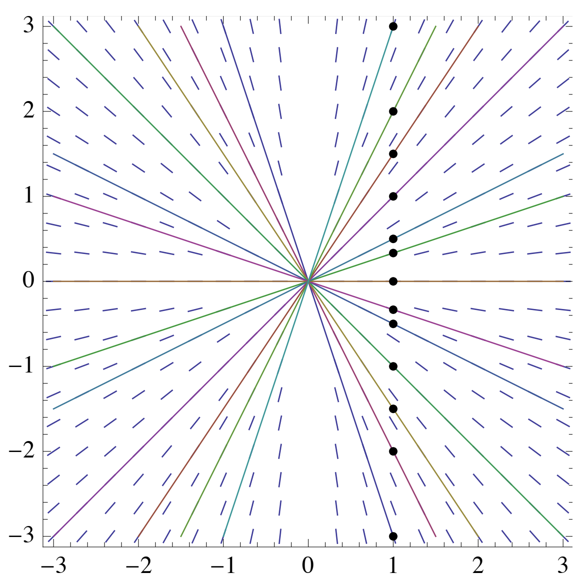
Solution: We see that the DE is already in normal form, with $f(x,y) = \sqrt[3]{y} = y^{1/3}$. We then examine this function and its derivative $f_y(x,y) = \frac{1}{3}y^{-2/3}$ around $x = 0$ and $y = 0$. While f is continuous everywhere, f_y is not even defined for $y = 0$. Hence, Theorem 1 does not apply, so neither existence nor uniqueness is guaranteed for this IVP.

By inspection, we can see that $y = 0$ is actually a solution to this DE passing through the initial condition $(0,0)$, so we have verified existence empirically. Since the DE is a fairly simple separable equation, we could probably solve it explicitly for y in terms of x, but we are not required to for this problem. ■

1.3.28. Verify that if k is a constant, then the function $y(x) \equiv kx$ satisfies the differential equation $xy' = y$. Construct a slope field and several of these straight-line solution curves. Then determine (in terms of a and b) how many different solutions the initial value problem $xy' = y$, $y(a) = b$ has—one, none, or infinitely many.

Solution: We first check that $y = kx$ satisfies the DE. For these solutions, $y' = k$, so $xy' = x(k) = kx = y$, as desired.

Putting the DE in normal form, $y' = y/x$. Letting $f(x,y) = y/x$, we plot a slope field with slopes given by f and some solutions $y = kx$:



Note that $f(x,y)$ is not defined for $x = 0$. We note that f and $f_y(x,y) = 1/x$ are continuous for $x \neq 0$, so the hypotheses of Theorem 1 are satisfied at all points in this region. Hence, there is a unique solution to this DE going through each (a,b) with $a \neq 0$, namely the line $y = bx/a$.

For $a = 0$, we see that all the linear solutions intersect at $(0,0)$, so there are an infinite number of solutions for the IC $(a,b) = (0,0)$. Conversely, no lines go through $(0,b)$ with $b \neq 0$, so there are no solutions to the IVP for that IC. ■

1.5.1. Find the general solution to the DE $y' + y = 2$. Find the particular solution satisfying the initial condition $y(0) = 0$.

Solution: We recognize this as a linear DE with $P(x) = 1$ and $Q(x) = 2$. We then multiply the DE by the integrating factor $\mu(x) = e^{\int P(x)\,dx} = e^x$ to get

$$(e^x y)' = e^x y' + e^x y = 2e^x.$$

Integrating, $e^x y = \int 2e^x\,dx = 2e^x + C$. Isolating y, $y = 2 + Ce^x$, C any real number, so this is the general solution to the DE.

Applying the initial condition, $0 = y(0) = 2 + Ce^0 = 2 + C$. Thus, $C = -2$, so $y = 2 - 2e^x$ is the particular solution. ∎

1.5.2. Find the general solution to the DE $y' - 2y = 3e^{2x}$. Find the particular solution satisfying the initial condition $y(0) = 0$.

Solution: We recognize this DE as being linear with $P(x) = -2$. Then $\int P(x)\,dx = -2x$, so an integrating factor is $\mu(x) = e^{-2x}$. Multiplying the DE by this $\mu(x)$, we have

$$(e^{-2x}y)' = 3e^{2x}e^{-2x} = 3,$$

and integrating yields $e^{-2x}y = 3x + C$. Hence, $y = 3xe^{2x} + Ce^{2x}$ is the general solution.

Applying the initial condition, $0 = y(0) = 3(0)e^0 + Ce^0 = C$, so $C = 0$. Thus, $y = 3xe^{2x}$ is the particular solution for this IVP. ∎

1.5.12. Find the general solution to the DE $xy' + 3y = 2x^5$. Find the particular solution satisfying the initial condition $y(2) = 1$.

Solution: Normalizing the DE, it becomes $y' + \frac{3}{x}y = 2x^4$. Then $P(x) = \frac{3}{x}$, so $\int P(x)\,dx = 3\ln|x|$, and an integrating factor is $\mu(x) = e^{3\ln|x|} = |x^3|$. In fact, we choose to take $\mu(x) = x^3$. Multiplying this through the normalized form of the equation,

$$(x^3y)' = x^3y' + 3x^2y = 2x^7.$$

Integrating, $x^3y = \int 2x^7\,dx = \frac{1}{4}x^8 + C$. Isolating y, $y = \frac{1}{4}x^5 + Cx^{-3}$.

Applying the IC, $1 = y(2) = \frac{1}{4}2^5 + C2^{-3} = 8 + C/8$. Then $C/8 = -7$, so $C = -56$, and the particular solution to the IVP is $y = \frac{1}{4}x^5 - 56x^{-3}$. ∎

1.5.14. Find the general solution to the DE $xy' - 3y = x^3$. Find the particular solution satisfying the initial condition $y(1) = 10$.

Solution: Normalizing the DE by dividing by x, we have $y' - \frac{3}{x}y = x^2$. Then $P(x) = -\frac{3}{x}$, so $\int P(x)\,dx = -3\ln|x|$, and an integrating factor is $e^{-3\ln|x|} = |x|^{-3}$. Actually, we take $\mu(x) = x^{-3} = 1/x^3$ as our integrating factor for convenience. Multiplying our normalized equation by this $\mu(x)$, we have

$$\left(\frac{1}{x^3}y\right)' = \frac{1}{x^3}y' - \frac{3}{x^4}y = x^2\frac{1}{x^3} = \frac{1}{x}.$$

Integrating, $y/x^3 = \ln x + C$, so $y = x^3 \ln x + Cx^3$ is the general solution to the DE.

Applying the initial condition, $10 = y(1) = 1^3 \ln 1 + C1^3 = 0 + C = C$, so $C = 10$. Thus, the particular solution to the IVP is $y = x^3 \ln x + 10x^3$. ∎

1.5.22. Find the general solution to the DE $y' = 2xy + 3x^2e^{x^2}$. Find the particular solution satisfying the initial condition $y(0) = 5$.

Solution: The DE is already given in normal form, so we move the y term to the left-hand side to get $y' - 2xy = 3x^2e^{x^2}$. Then this DE is linear with $P(x) = -2x$, so $\int P(x)\,dx = -x^2$, and an integrating factor is $\mu(x) = e^{-x^2}$. Multiplying the DE on both sides by this function,

$$(e^{-x^2}y)' = e^{-x^2}y' - 2xy = 3x^2e^{x^2}e^{-x^2} = 3x^2.$$

Then integrating yields $e^{-x^2}y = \int 3x^2\,dx = x^3 + C$. Isolating y gives the general solution $y = x^3e^{x^2} + Ce^{x^2}$.

Applying the initial condition, $5 = y(0) = 0^3e^0 + Ce^0 = C$, so $C = 5$, and $y = (x^3 + 5)e^{x^2}$ is the particular solution to this IVP. ∎

1.5.36. A tank initially contains 60 gallons of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus, the tank is empty after exactly 1 hour.

(a) Find the amount of salt in the tank after t minutes.

(b) What is the maximum amount of salt ever in the tank?

Solution (a): We produce a model for this system. Let $x(t)$ denote the amount of salt in the tank at time t, t in minutes since the brine starts flowing into the tank. The flow rate in is $r_i = 2$ gal/min, and the flow rate out is $r_o = 3$ gal/min, so the volume in the tank at time t is $V(t) = 60 + (2-3)t = 60 - t$, in gallons.

The concentration of salt in the tank (and hence of the solution flowing out of the tank) is then $c_o(t) = x(t)/V(t) = \frac{1}{60-t}x(t)$. Since the incoming concentration is $c_i = 1$ lb/gal, the net rate of change of the salt is

$$\frac{dx}{dt} = r_i c_i - r_o c_o = (2)(1) - (3)\frac{1}{60-t}x(t) = 2 - \frac{3}{60-t}x(t),$$

which is a linear DE. Rearranging it so that $x(t)$ and $x'(t)$ are on the left-hand side, we have $x' + \frac{3}{60-t}x = 2$. Then $P(t) = \frac{3}{60-t}$, so $\int P(t)\,dt = -3\ln|60-t|$, with the minus sign coming from the chain rule with $60-t$. Hence, an integrating factor is $e^{-3\ln|60-t|} = |60-t|^{-3}$. We choose to take $\mu(t) = (60-t)^{-3}$, without the absolute values, since we expect solutions only for $0 \leq t \leq 60$.

Multiplying the DE by this integrating factor, we have

$$\left(\frac{1}{(60-t)^3}x\right)' = \frac{1}{(60-t)^3}x' + \frac{3}{(60-t)^4}x = \frac{2}{(60-t)^3}.$$

Integrating, $\frac{1}{(60-t)^3}x = \int \frac{2}{(60-t)^3}\,dt = \frac{1}{(60-t)^2} + C$, so $x = 60 - t + C(60-t)^3$.

Finally, we note that there is also an initial condition: $x(0) = 0$, since the tank starts with no salt. Thus, $0 = x(0) = 60 - 0 + C(60-0)^3 = 60 + 60^3C$, so $C = 1/60^2$, and the total amount of salt in the tank at time t is

$$x(t) = 60 - t - \frac{1}{3600}(60-t)^3.$$

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Solution (b): We find the maximum value of $x(t)$ on $[0, 60]$. Computing $x'(t)$,

$$x'(t) = -1 - \frac{1}{3600}(3)(-1)(60-t)^2 = -1 + \frac{1}{1200}(60-t)^2.$$

At a local extremum, $x'(t) = 0$, since $x'(t)$ exists everywhere on this interval. Then $0 = -1 + \frac{1}{1200}(60-t)^2$, so $(60-t)^2 = 1200$, and $t = 60 - 20\sqrt{3}$. Finally,

$$x(60 - 20\sqrt{3}) = 20\sqrt{3} - \frac{1}{3600}(20\sqrt{3})^3 = 20\sqrt{3} - \frac{1}{3}20\sqrt{3} = \frac{40}{3}\sqrt{3} \approx 23.09 \text{ lbs}.$$

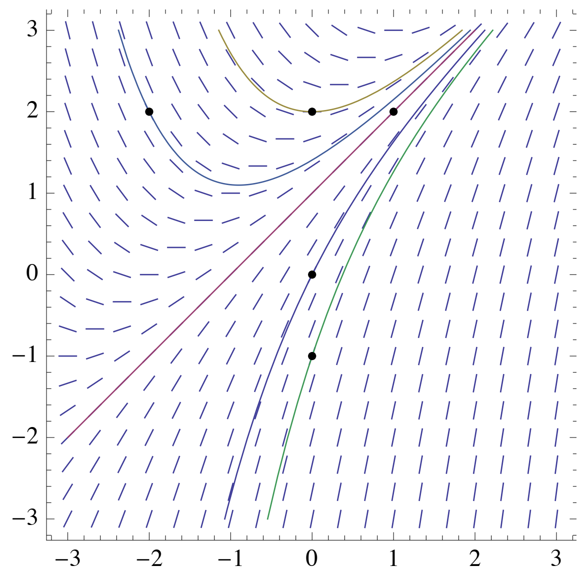
Since $x(0) = x(60) = 0$, this is the maximum amount of salt in the tank. ■

Extra Problem #1. Consider the DE $y' = x + 2 - y$.

(a) Plot a slope field for this DE in the region $-3 \leq x \leq 3$, $-3 \leq y \leq 3$. Sketch solution curves through the points $(0,0)$, $(0,2)$, and $(1,2)$. Sketch at least two other solution curves of your choice.

(b) Find the general solution to this DE. Find a particular solution satisfying the initial condition $y(1) = 2 - \frac{2}{e}$. Sketch it on your slope field.

Solution (a): Here is the slope field, with solutions through $(0,0)$, $(0,2)$, and $(1,2)$, as well as $(0,-1)$ and $(-2,2)$:



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Solution (b): Writing the DE as $y' + y = x + 2$, we see it is linear. We multiply by the integrating factor $\mu(x) = e^x$, so $(e^x y)' = xe^x + 2e^x$. Integrating and applying integration by parts for the right-hand side,

$$e^x y = (xe^x - e^x) + 2e^x + C = (x+1)e^x + C,$$

so $y = x + 1 + Ce^{-x}$ is the general solution to the DE.

We solve for C with the IC $y(1) = 2 - \frac{2}{e} = 1 + 1 + Ce^{-1}$. Then $C = -2$, so $y = x + 1 - 2e^{-x}$. In fact, $y(0) = 1 - 2 = -1$, so this is the curve through $(0,-1)$, which we already sketched above. ■