Homework #6 Solutions

Problems

- Section 3.1: 26, 34, 40, 46
- Section 3.2: 2, 8, 10, 14, 18, 24, 30

3.1.26. Determine whether the functions $f(x) = 2\cos x + 3\sin x$ and $g(x) = 3\cos x - 2\sin x$ are linearly dependent or linearly independent on the real line.

Solution: We compute the Wronskian of these functions:

$$\begin{aligned} W(f,g) &= \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} 2\cos x + 3\sin x & 3\cos x - 2\sin x \\ -2\sin x + 3\cos x & -3\sin x - 2\cos x \end{vmatrix} \\ &= (2\cos x + 3\sin x)(-3\sin x - 2\cos x) - (3\cos x - 2\sin x)(-2\sin x + 3\cos x) \\ &= (-12\cos x\sin x - 9\sin^2 x - 4\cos^2 x) + (12\cos x\sin x - 9\cos^2 x - 4\sin^2 x) \\ &= 13(\sin^2 x + \cos^2 x) = 13 \end{aligned}$$

Since the Wronskian is the constant function 13, which is not the 0 function, these functions are linearly independent on the real line (and in fact on any subinterval of the real line). ∎

3.1.34. Find the general solution of the DE $y'' + 2y' - 15y = 0$.

Solution: Guessing the solution $y = e^{rx}$, we obtain the characteristic equation $r^2 - 2r + 15 = 0$, which factors as $(r-5)(r+3) = 0$. Therefore, $r = 5$ and $r = -3$ are roots, so $y_1 = e^{5x}$ and $y_2 = e^{-3x}$ are solutions. Furthermore, by Theorem 5 in §3.1, the general solution to the DE is

$$y = c_1 e^{5x} + c_2 e^{-3x}.$$ ∎

3.1.40. Find the general solution of the DE $9y'' - 12y' + 4y = 0$.

Solution: As above, the characteristic equation for the DE is $9r^2 - 12r + 4 = 0$, which factors as $(3r-2)^2 = 0$. Therefore, this equation has a double root at $r = 2/3$. By Theorem 6 in §3.1, the general solution is then

$$y = c_1 x e^{2x/3} + c_2 e^{2x/3}.$$ ∎

3.1.46. Find a homogeneous second-order DE $ay'' + by' + cy = 0$ with general solution $y = c_1 e^{10x} + c_2 e^{100x}$.

Solution: This constant coefficient DE must have $y_1 = e^{10x}$ and $y_2 = e^{100x}$ as solutions, so we expect $r - 10$ and $r - 100$ to be factors of its characteristic polynomial. Then we may take

$$ar^2 + br + c = a(r-10)(r-100) = a(r^2 - 110r + 1000),$$

so taking $a = 1$, we have the corresponding DE $y'' - 110y' + 1000y = 0$. ∎

3.2.2. Show directly that the functions $f(x) = 5$, $g(x) = 2 - 3x^2$, and $h(x) = 10 + 15x^2$ are linearly dependent on the real line.

Solution: We find a nontrivial linear combination $c_1 f + c_2 g + c_3 h$ of these functions identically equal to 0. Since all 3 functions are polynomials in x, the function is 0 exactly when the coefficients on all the powers of x are 0. Since

$$\begin{aligned} c_1 f + c_2 g + c_3 h &= c_1(5) + c_2(2 - 3x^2) + c_3(10 + 15x^2) \\ &= (5c_1 + 2c_2 + 10c_3) + (-3c_2 + 15c_3)x^2, \end{aligned}$$

we require that $5c_1 + 2c_2 + 10c_3 = 0$ and $-3c_2 + 15c_3 = 0$. From the second equation, $c_2 = 5c_3$. Substituting this into the first,

$$5c_1 + 2c_2 + 10c_3 = 5c_1 + 2(5c_3) + 10c_3 = 5c_1 + 20c_3 = 0.$$

Then $c_1 = -4c_3$, and there are no more constraints on the c_i. Choosing to set $c_3 = 1$, $c_1 = -4$ and $c_2 = 5$. We check that this nontrivial linear combination of functions is 0:

$$(-4)(5) + (5)(2 - 3x^2) + (1)(10 + 15x^2) = -20 + 10 - 15x^2 + 10 + 15x^2 = 0.$$

Rearranging this equation, we can express any single one of these functions as a linear combination of the other two: for example, $10 + 15x^2 = 4(5) - 5(10 - 3x^2)$. ∎

3.2.8. Use the Wronskian to prove that the functions $f(x) = e^x$, $g(x) = e^{2x}$, and $h(x) = e^{3x}$ are linearly independent on the real line.

Solution: We compute $W(f, g, h)$:

$$\begin{aligned} W(f,g,h) &= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} \\ &= e^x \left(\begin{vmatrix} 2e^{2x} & 3e^{3x} \\ 4e^{2x} & 9e^{3x} \end{vmatrix} - \begin{vmatrix} e^{2x} & e^{3x} \\ 4e^{2x} & 9e^{3x} \end{vmatrix} + \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} \right) \\ &= e^x e^{2x} e^{3x} \left((2 \cdot 9 - 4 \cdot 3) - (1 \cdot 9 - 1 \cdot 4) + (1 \cdot 3 - 1 \cdot 2) \right) \\ &= e^{6x}(6 - 5 + 1) = 2e^{6x}. \end{aligned}$$

Since $W(x) = 2e^{6x}$, which is not zero on the real line (and in fact nowhere 0), these three functions are linearly independent. ∎

3.2.10. Use the Wronskian to prove that the functions $f(x) = e^x$, $g(x) = x^{-2}$, and $h(x) = x^{-2}\ln x$ are linearly independent on the interval $x > 0$.

Solution: We compute $W(f, g, h)$. First, we compute derivatives of h:

$$h'(x) = -2x^{-3}\ln x + x^{-2}\frac{1}{x} = (1 - 2\ln x)x^{-3}$$
$$h''(x) = (-3)(1 - 2\ln x)x^{-4} + \frac{-2}{x}x^{-3} = (6\ln x - 5)x^{-4}$$

Plugging these into the Wronskian, we have

$$W(f, g, h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} e^x & x^{-2} & x^{-2}\ln x \\ e^x & -2x^{-3} & (1 - 2\ln x)x^{-3} \\ e^x & 6x^{-4} & (6\ln x - 5)x^{-4} \end{vmatrix}.$$

Rather than expand this directly, we make use of some additional properties of the determinant. One of these is that the determinant is unchanged if a multiple of one column is added to or subtracted from a different column. We subtract $\ln x$ times the second column from the third to cancel the $\ln x$ terms there:

$$W(f, g, h) = \begin{vmatrix} e^x & x^{-2} & 0 \\ e^x & -2x^{-3} & x^{-3} \\ e^x & 6x^{-4} & -5x^{-4} \end{vmatrix}$$

Using another property of the determinant, we factor the scalar e^x out of the first column, so that it multiplies the determinant of the remaining matrix:

$$W(f, g, h) = e^x \begin{vmatrix} 1 & x^{-2} & 0 \\ 1 & -2x^{-3} & x^{-3} \\ 1 & 6x^{-4} & -5x^{-4} \end{vmatrix}$$

With these simplifications, we expand along the first row, which conveniently contains a 0 entry:

$$\begin{aligned} W(f, g, h) &= e^x \left(\begin{vmatrix} -2x^{-3} & x^{-3} \\ 6x^{-4} & -5x^{-4} \end{vmatrix} - x^{-2} \begin{vmatrix} 1 & x^{-3} \\ 1 & -5x^{-4} \end{vmatrix} + 0 \right) \\ &= e^x \left(10x^{-7} - 6x^{-7} - x^{-2}(-5x^{-4} - x^{-3}) \right) \\ &= e^x x^{-7}(x^2 + 5x + 4) = \frac{e^x(x+1)(x+4)}{x^7}. \end{aligned}$$

This function is defined and continuous for all $x > 0$. Furthermore, none of the factors in its numerator is 0 for $x > 0$, so it is in fact nowhere 0 on this interval. Since their Wronskian is not identically 0, these functions are linearly independent on this interval. ∎

3.2.14. Find a particular solution to the DE $y^{(3)} - 6y'' + 11y' - 6y = 0$ matching the initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 3$ that is a linear combination of $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$.

Solution: We let $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$. Then $y' = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}$ and $y'' = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}$, so evaluating these functions at $x = 0$ and matching them to the initial conditions, we obtain the linear system

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 + 2c_2 + 3c_3 &= 0 \\ c_1 + 4c_2 + 9c_3 &= 3 \end{aligned}$$

We solve this linear system by row reduction of an augmented matrix to echelon form:

$$\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 4 & 9 & 3 \end{array}\right] &\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 8 & 3 \end{array}\right] & R_2 \leftarrow R_2 - R_1, R_3 \leftarrow R_3 - R_1 \\
&\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 3 \end{array}\right] & R_3 \leftarrow R_3 - 3R_2 \\
&\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right] & R_3 \leftarrow \frac{1}{2}R_3, R_2 \leftarrow R_2 - 2R_3, R_1 \leftarrow R_1 - R_3 \\
&\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right] & R_1 \leftarrow R_1 - R_2
\end{aligned}$$

Then $c_1 = c_3 = \frac{3}{2}$ and $c_2 = -3$, so $y = \frac{3}{2}e^x - 3e^{2x} + \frac{3}{2}e^{3x}$ is the solution to the IVP. ■

3.2.18. Find a particular solution to the DE $y^{(3)} - 3y'' + 4y' - 2y = 0$ matching the initial conditions $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$ that is a linear combination of $y_1 = e^x$, $y_2 = e^x \cos x$, and $y_3 = e^x \sin x$.

Solution: We let $y = c_1e^x + c_2e^x \cos x + c_3e^x \sin x$. Then

$$\begin{aligned} y' &= c_1e^x + c_2e^x(\cos x - \sin x) + c_3e^x(\sin x + \cos x) \\ y'' &= c_1e^x - 2c_2e^x \sin x + 2c_3e^x \cos x \end{aligned}$$

Evaluating these functions at $x = 0$ and matching them to the initial conditions, we obtain the linear system

$$\begin{aligned} c_1 + c_2 \qquad &= 1 \\ c_1 + c_2 + \ \ c_3 &= 0 \\ c_1 + \qquad 2c_3 &= 0 \end{aligned}$$

By the third equation, $c_1 = -2c_3$. Substituting this into the second, $c_2 - c_3 = 0$, so $c_2 = c_3$. Finally, in the first equation, $-2c_3 + c_3 = 1$, so $c_3 = -1$, $c_2 = -1$, and $c_1 = -2(-1) = 2$. Then $y = 2e^x - e^x \cos x - e^x \sin x = e^x(2 - \cos x - \sin x)$ is the solution to the IVP. ∎

3.2.24. The nonhomogeneous DE $y'' - 2y' + 2y = 2x$ has the particular solution $y_p = x + 1$ and the complementary solution $y_c = c_1 e^x \cos x + c_2 e^x \sin x$. Find a solution satisfying the initial conditions $y(0) = 4$, $y'(0) = 8$.

Solution: The general solution to this DE is of the form

$$y = y_p + y_c = x + 1 + c_1 e^x \cos x + c_2 e^x \sin x.$$

Then

$$y' = 1 + c_1 e^x(\cos x - \sin x) + c_2 e^x(\sin x + \cos x).$$

Evaluating at $x = 0$ and applying the initial conditions, $y(0) = 1 + c_1 = 4$ and $y'(0) = 1 + c_1 + c_2 = 8$. Then $c_1 = 3$ and $c_2 = 4$, so the solution to the IVP is

$$y = x + 1 + e^x(3\cos x + 4\sin x).$$

∎

3.2.30. Verify that $y_1 = x$ and $y_2 = x^2$ are linearly independent solutions on the entire real line of the equation $x^2 y'' - 2xy' + 2y = 0$, but that $W(x, x^2)$ vanishes at $x = 0$. Why do these observations not contradict part (b) of Theorem 3?

Solution: We first check that these are solutions:

$$x^2 y_1'' - 2xy_1' + 2y_1 = x^2(0) - 2x(1) + 2(x) = x(-2 + 2) = 0$$
$$x^2 y_2'' - 2xy_2' + 2y_2 = x^2(2) - 2x(2x) + 2(x^2) = x^2(2 - 4 + 2) = 0$$

We then compute their Wronskian:

$$W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x(2x) - 1(x^2) = x^2.$$

This function is not identically 0, so the two functions x and x^2 are linearly independent on the real line, but it is 0 at precisely $x = 0$.

We note that Theorem 3 applies only to normalized homogeneous linear DEs. Normalizing this DE, we obtain

$$y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 0,$$

the coefficient functions of which are continuous for $x \neq 0$. Thus, any interval on which the theorem applies does not include $x = 0$, the only point at which $W(x) = 0$, so $W(x, x^2)$ is nonzero on every such interval. This is consistent with the linear independence of the solutions x and x^2. ∎