Homework #7 Solutions

Problems

- Section 3.3: 4, 18, 22, 24, 34, 40
- Section 3.4: 4, 12abc, 16, 18, 22. Omit the graphing part on problems 16 and 18.

3.3.4. Find the general solution to the differential equation $2y'' - 7y' + 3y = 0$.

Solution: We determine that the characteristic equation for this linear polynomial is $2r^2 - 7r + 3 = 0$, which factors as $(2r-1)(r-3) = 0$. Thus, the roots are $r = 1/2$ and $r = 3$, so the general solution is $y = c_1 e^{x/2} + c_2 e^{3x}$. ∎

3.3.18. Find the general solution to the differential equation $y^{(4)} = 16y$.

Solution: Writing the DE as $y^{(4)} - 16y = 0$, we see that its characteristic equation is $r^4 - 16 = 0$. Since this is a difference of squares, it factors as

$$(r^2-4)(r^2+4) = (r-2)(r+2)(r^2+4) = 0.$$

Therefore, it has the roots $r = 2$ and $r = -2$ from the linear factors, and the $r^2 + 4$ factor has pure imaginary roots $r = \pm\sqrt{-4} = \pm 2i$. From the real roots, we have the solutions e^{2x} and e^{-2x}, while we get the trigonometric functions $\cos 2x$ and $\sin 2x$ from the pure imaginary roots. Thus, the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x.$$ ∎

3.3.22. Solve the IVP $9y'' + 6y' + 4y = 0$, $y(0) = 3$, $y'(0) = 4$.

Solution: We first find the general solution to the homogeneous linear DE. Since its characteristic equation is $9r^2 + 6r + 4 = 0$, which has roots

$$r = \frac{-6 \pm \sqrt{6^2 - 4(4)(9)}}{2 \cdot 9} = \frac{-1 \pm \sqrt{-3}}{3} = -\frac{1}{3} \pm \frac{1}{\sqrt{3}} i,$$

the general solution is the linear combination

$$y = c_1 e^{-x/3} \cos \frac{x}{\sqrt{3}} + c_2 e^{-x/3} \sin \frac{x}{\sqrt{3}}.$$

We use the product rule to compute its derivative; after factoring out $e^{-x/3}$, this is

$$y' = c_1 e^{-x/3}\left(-\frac{1}{3}\cos\frac{x}{\sqrt{3}} - \frac{1}{\sqrt{3}}\sin\frac{x}{\sqrt{3}}\right) + c_2 e^{-x/3}\left(-\frac{1}{3}\sin\frac{x}{\sqrt{3}} + \frac{1}{\sqrt{3}}\cos\frac{x}{\sqrt{3}}\right).$$

Evaluating at $x = 0$ and applying the initial conditions, and noting that the sin terms vanish,

$$y(0) = c_1 = 3, \qquad y'(0) = -\frac{c_1}{3} + \frac{c_2}{\sqrt{3}} = 4.$$

Then $c_1 = 3$, and $c_2 = \sqrt{3}(4 + c_1/3) = 5\sqrt{3}$, so the solution to the IVP is

$$y = 3e^{-x/3}\cos\frac{x}{\sqrt{3}} + 5\sqrt{3}e^{-x/3}\sin\frac{x}{\sqrt{3}}.$$

■

3.3.24. Solve the IVP $2y^{(3)} - 3y'' - 2y' = 0$, $y(0) = 1$, $y'(0) = -1$, $y''(0) = 3$.

Solution: The characteristic equation for this DE is $2r^3 - 3r^2 - 2r = 0$, which factors as

$$r(2r^2 - 3r - 2) = r(2r + 1)(r - 2) = 0.$$

Then $r = 0$, $r = -1/2$, and $r = 2$ are its (distinct real) roots, so the general solution is

$$y = c_1 + c_2 e^{-x/2} + c_3 e^{2x}.$$

Its first and second derivatives are

$$y' = -\frac{1}{2}c_2 e^{-x/2} + 2c_3 e^{2x}, \qquad y'' = \frac{1}{4}c_2 e^{-x/2} + 4c_3 e^{2x}.$$

Evaluating at $x = 0$ and applying the initial conditions, we have the linear system

$$c_1 + c_2 + c_3 = 1, \qquad -\frac{1}{2}c_2 + 2c_3 = -1, \qquad \frac{1}{4}c_2 + 4c_3 = 3.$$

From the third equation, $c_2 = 12 - 16c_3$. Substituting this into the second, $-\frac{1}{2}(12 - 16c_3) + 2c_3 = -1$, so $10c_3 = 5$, and $c_3 = 1/2$. Then $c_2 = 12 - 8 = 4$, so $c_1 = 1 - c_2 - c_3 = -7/2$. Thus, the solution to the IVP is $y = -\frac{7}{2} + 4e^{-x/2} + \frac{1}{2}e^{2x}$. ■

3.3.34. One solution to the DE $3y^{(3)} - 2y'' + 12y' - 8y = 0$ is $y = e^{2x/3}$. Find the general solution.

Solution: The characteristic equation for the DE is $3r^3 - 2r^2 + 12r - 8 = 0$. Since $r = 2/3$ is a root, we expect $r - 2/3$ to be a linear factor of this polynomial. In fact, $3(r - 2/3) = 3r - 2$ is seen to be a linear factor, so that this polynomial factors as

$$(3r - 2)(r^2 + 4) = 0.$$

Thus, the two other roots are $r = \pm 2i$, from the $r^2 + 4$ factor, so the general solution is

$$c_1 e^{2x/3} + c_2 \cos 2x + c_3 \sin 2x.$$

■

3.3.40. Find a linear homogeneous constant-coefficient equation with general solution $y = Ae^{2x} + B\cos 2x + C\sin 2x$.

Solution: Since the general solution contains both e^{2x} and a $\cos 2x$-$\sin 2x$ pair, the original DE should have roots $r = 2$ and $r = \pm 2i$. Furthermore, its general solution has three independent parameters, so it should be a third-order DE. Hence, its characteristic equation is

$$(r-2)(r-2i)(r+2i) = (r-2)(r^2+4) = r^3 - 2r^2 + 4r - 8 = 0,$$

which comes from the linear homogeneous DE $y^{(3)} - 2y'' + 4y' - 8y = 0$. (Scalar multiples of this DE have the same solutions.) ∎

3.4.4. A body with mass 250 g is attached to the end of a spring that is stretched 25 cm by a force of 9 N. At time $t = 0$ the body is pulled 1 m to the right, stretching the spring, and set in motion with an initial velocity of 5 m/s to the left.

(a) Find $x(t)$ in the form $C\cos(\omega_0 t - \alpha)$.

(b) Find the amplitude and period of motion of the body.

Solution (a): We normalize the constants to mks SI units, so $m = 0.25$ kg and $k = 9/0.25 = 36$ N/m. Then the circular frequency is $\omega_0 = \sqrt{k/m} = \sqrt{36/0.25} = 12$ rad/s, so the general solution for the motion of the body is

$$x(t) = A\cos 12t + B\sin 12t, \qquad x'(t) = -12A\sin 12t + 12B\cos 12t.$$

Evaluating at $t = 0$, $x(0) = A = 1$ and $x'(0) = 12B = -5$ (since the initial velocity is rightward). Then $A = 1$ and $B = -5/12$. Computing C, $C^2 = A^2 + B^2 = \frac{12^2+5^2}{12^2} = (\frac{13}{12})^2$, so $C = \frac{13}{12}$.

The phase angle α has $\tan\alpha = B/A = -\frac{5}{12}$, but must satisfy $C\cos\alpha = A > 0$ and $C\sin\alpha = B < 0$ from the above values of A and B. Fortunately, since its cosine must be positive, we may then pick α from the principal branch of arctan, so $\alpha = \arctan(-5/12) = -\arctan\frac{5}{12} \approx -0.395$. (Of course, we may also add an arbitrary multiple of 2π to this angle, so $2\pi - \arctan\frac{5}{12} \approx 5.888$ is also a correct answer.) Thus,

$$x(t) = \frac{13}{12}\cos\left(12t + \arctan\frac{5}{12}\right).$$

∎

Solution (b): The amplitude is the constant $C = \frac{13}{12}$, and the period is $T = \frac{2\pi}{12} = \frac{\pi}{6}$ s. ∎

3.4.12abc. Assume that the earth is a solid sphere of uniform density, with mass M and radius $R = 3960$ miles. For a particle of mass m within the earth at a distance r from the center of the earth, the gravitational force attracting m toward the center is $F_r = -GM_r m/r^2$, where M_r is the mass of the part of the earth within a sphere of radius r.

(a) Show that $F_r = -GMmr/R^3$.

(b) Now suppose that a small hole is drilled straight through the center of the earth, thus connecting two antipodal paints on its surface. Let a particle of mass m be dropped at time $t = 0$ into this hole with initial speed zero, and let $r(t)$ be its distance from the center of the earth at time t. Conclude from Newton's second law and part (a) that $r''(t) = -k^2 r(t)$, where $k^2 = GM/R^3 = g/R$.

(c) Take $g = 32.2$ ft/s^2, and conclude from part (b) that the particle undergoes simple harmonic motion back and forth between the ends of the hole, with a period of about 84 min.

Solution (a): The volume of the Earth (assumed to be a perfect sphere) is $V = \frac{4\pi}{3}R^3$, so the density of the Earth is $\rho = \frac{M}{V} = \frac{M}{\frac{4\pi}{3}R^3}$. The mass of the radius-r portion of the Earth is then $M_r = \rho \cdot \frac{4\pi}{3}r^3 = M\frac{r^3}{R^3}$. Hence, the force of gravity on the mass m at its surface is

$$F_r = -\frac{Gm}{r^2}M_r = -\frac{Gm}{r^2}M\frac{r^3}{R^3} = -\frac{GMmr}{R^3}.$$

■

Solution (b): From Newton's second law, $F = ma$. Since the position of the particle is given by $r(t)$, the acceleration is its second derivative, $r''(t)$. The only force on the body is gravity, so from part (a), we have the DE.

$$-\frac{GMmr}{R^3} = mr''(t).$$

Dividing out the m and collecting the r terms, this is

$$r'' + \frac{GM}{R^3}r = 0,$$

which gives simple harmonic motion with circular frequency $\omega_0 = \sqrt{\frac{GM}{R^3}}$. We also note that when $r = R$, the gravitational acceleration g is given by $\frac{GMR}{R^3} = \frac{GM}{R^2}$, so this constant is also $\frac{g}{R}$. ■

Solution (c): Since $\omega_0 = \sqrt{\frac{g}{R}}$, the period of oscillation is $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{R}{g}}$. Then, in fps units,

$$T = 2\pi\sqrt{\frac{3960 \cdot 5280}{32.2}} \approx 5063 \text{ s} \approx 84.4 \text{ min}.$$

■

3.4.16. A mass $m = 3$ is attached to both a spring with spring constant $k = 63$ and a dashpot with damping constant $c = 30$. The mass is set in motion with initial position $x_0 = 2$ and initial velocity $v_0 = 2$. Find the position function $x(t)$ and determine whether the motion is overdamped, critically damped, or underdamped. If it is underdamped, write the position function in the form $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha_1)$. Also, find the undamped position function $u(t) = C_0\cos(\omega_0 t - \alpha_0)$ that would result if the mass on the spring were set in motion with the same initial position and velocity but wth the dashpot disconnected ($c = 0$).

Solution: We first find that $\omega_0^2 = \frac{k}{m} = 21$, and $p = \frac{c}{2m} = \frac{30}{2\cdot 3} = 5$. Then $p^2 = 25$, which is larger than ω_0^2, so the motion is overdamped. The roots of the DE are $r = -p \pm \sqrt{p^2 - \omega_0^2} = -5 \pm 2$, or $r = -7$ and $r = -3$. Hence, the general solution is

$$x(t) = c_1 e^{-7t} + c_2 e^{-3t}, \qquad v(t) = x'(t) = -7c_1 e^{-7t} - 3c_2 e^{-3t}.$$

Setting $t = 0$ and matching the initial conditions, $x(0) = c_1 + c_2 = 2$ and $v(0) = -7c_1 - 3c_2 = 2$. Solving for c_1 and c_2, $c_2 = 2 - c_1$, so $-7c_1 - 3(2 - c_1) = 2$, and $-4c_1 = 8$, Then $c_1 = -2$, so $c_2 = 4$, and the solution is

$$x(t) = 4e^{-3t} - 3e^{-7t}.$$

Removing the damping, we get simple harmonic motion with the frequency $\omega_0 = \sqrt{21}$, so the general solution is

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t, \qquad u'(t) = -A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t.$$

Then $u(0) = A = 2$ and $u'(0) = B\omega_0 = 2$, so $A = 2$ and $B = \frac{2}{\sqrt{21}}$. Then $u(t) = C\cos(\omega_0 t - \alpha)$ $C = \sqrt{A^2 + B^2} = \sqrt{4 + \frac{4}{21}} = 2\sqrt{\frac{22}{21}}$, and $\alpha = \arctan\frac{B}{A} = \arctan\frac{1}{\sqrt{21}}$. ∎

3.4.18. A mass $m = 2$ is attached to both a spring with spring constant $k = 50$ and a dashpot with damping constant $c = 12$. The mass is set in motion with initial position $x_0 = 0$ and initial velocity $v_0 = -8$. Find the position function $x(t)$ and determine whether the motion is overdamped, critically damped, or underdamped. If it is underdamped, write the position function in the form $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha_1)$. Also, find the undamped position function $u(t) = C_0\cos(\omega_0 t - \alpha_0)$ that would result if the mass on the spring were set in motion with the same initial position and velocity but wth the dashpot disconnected ($c = 0$).

Solution: We have that $\omega_0^2 = \frac{k}{m} = \frac{50}{2} = 25$, so $\omega_0 = 5$, and $p = \frac{c}{2m} = \frac{12}{2\cdot 2} = 3$, so the motion is underdamped. The pseudofrequency is $\omega_1 = \sqrt{\omega_0^2 - p^2} = \sqrt{25 - 9} = \sqrt{16} = 4$, so the general solution is

$$x(t) = c_1 e^{-3t}\cos 4t + c_2 e^{-3t}\sin 4t,$$
$$v(t) = c_1 e^{-3t}(-3\cos 4t - 4\sin 4t) + c_2 e^{-3t}(-3\sin 4t + 4\cos 4t)$$

At $t = 0$, $x(0) = c_1 = 0$. Then, using $c_1 = 0$, $v(0) = 4c_2 = -8$, so $c_2 = -2$. Hence, $x(t) = -2e^{-3t}\sin 4t$, which we convert to $Ce^{-pt}\cos(\omega_1 t - \alpha_1)$. We have that $C = 2$, and $A = -2 = 2\sin\alpha_1$, so $\alpha_1 = \frac{3\pi}{2}$ (taking an angle between 0 and 2π), so

$$x(t) = 2e^{-3t}\cos\left(4t - \frac{3}{2}\pi\right).$$

With $c = 0$, the solution is $u(t) = A\cos 5t + B\sin 5t$, with $u'(t) = -5A\sin 5t + 5B\cos 5t$; applying the initial conditions, $A = 0$ and $5B = -8$, so $u(t) = -\frac{8}{5}\sin 5t = \frac{8}{5}\cos(5t - \frac{3}{2}\pi)$. ■

3.4.22. A 12-lb weight (mass $m = 0.375$ slugs in fps units) is attached both to a vertically suspended spring that it stretches 6 inches and to a dashpot that provides 3 lb of resistance for every foot-per-second of velocity.

(a) If the weight is pulled down 1 foot below its static equilibrium position and then released from rest at time $t = 0$, find its position function $x(t)$.

(b) Find the frequency, time-varying amplitude, and phase angle of the motion.

Solution (a): The mass in fps units is $m = 0.375$, and the spring constant is $k = \frac{12 \text{ lb}}{0.5 \text{ ft}} = 24$ lb/ft. The circular frequency is given by $\omega_0^2 = \frac{k}{m} = 64$, so $\omega_0 = 8$ rad/s. The damping constant is $c = 3$ lb-s/ft, so $p = \frac{c}{2m} = 4$, and the system is therefore underdamped. Then $\omega_1 = \sqrt{\omega_0^2 - p^2} = \sqrt{48} = 4\sqrt{3}$, so the general solution is

$$x(t) = Ae^{-4t}\cos 4\sqrt{3}t + Be^{-4t}\sin 4\sqrt{3}t,$$
$$v(t) = Ae^{-4t}(-4\cos 4\sqrt{3}t - 4\sqrt{3}\sin 4\sqrt{3}t) + Be^{-4t}(-4\sin 4\sqrt{3}t + 4\sqrt{3}\cos 4\sqrt{3}t).$$

We measure the displacement vertically, considering a displacement downwards as being positive, since it corresponds to stretching the string further. Thus, at time $t = 0$ we have that $x(0) = 1$ and $v(0) = 0$. Then $A = 1$ and $-4A + 4\sqrt{3}B = 0$, so $B = \frac{4A}{4\sqrt{3}} = \frac{1}{\sqrt{3}}$. Thus, the solution is

$$x(t) = e^{-4t}\cos 4\sqrt{3}t + \frac{1}{\sqrt{3}}e^{-4t}\sin 4\sqrt{3}.$$

■

Solution (b): We reformulate our answer to part (a) in the form $Ce^{-pt}\cos(\omega_1 t - \alpha)$. Then $C^2 = A^2 + B^2 = 1 + \frac{1}{3} = \frac{4}{3}$, so $C = \frac{2}{\sqrt{3}}$, and the time-varying amplitude is $\frac{2}{\sqrt{3}}e^{-4t}$. From above, the pseudofrequency is $\omega_1 = 4\sqrt{3}$ rad/s. Finally, $\tan\alpha = \frac{B}{A} = \frac{1}{\sqrt{3}}$, with α in Quadrant I so that both A and B are positive. Thus, the phase angle is $\alpha = \frac{\pi}{6}$. ■