Homework #9 Solutions

Problems

- Section 3.6: 4, 8, 12, 18, 28, with modified graphing directions below:
  - On #4, omit the graph.
  - On #8, graph $x_{\text{sp}}(t)$ and $\tilde{F}(t) = \frac{F(t)}{m\omega^2}$ (which has units of length, unlike $F(t)/m\omega$).
  - On #12, graph both $x_{\text{sp}}(t)$ and $x(t) = x_{\text{sp}}(t) + x_{\text{tr}}(t)$.
- Section 4.1: 2, 8, 24

3.6.4. Express the solution $x(t)$ to the IVP $x'' + 25x = 90\cos 4t$, $x(0) = 0$, $x'(0) = 90$ as the sum of two oscillations.

Solution: We first find the complementary solution $x_c(t)$ to this nonhomogeneous DE. Since it is a simple harmonic oscillation system with $m = 1$ and $k = 25$, the circular frequency is $\omega_0 = \sqrt{25} = 5$, and

$$x_c(t) = c_1 \cos 5t + c_2 \sin 5t.$$

Since the forcing term has frequency $\omega = 4$, which is not equal to ω_0, we expect a steady state solution $x_p(t)$ of the form $A\cos 4t + B\sin 4t$. Differentiating twice, we see that $x_p'' = -16x_p$, so we obtain the equation

$$9A\cos 4t + 9B\sin 4t = 90\cos 4t.$$

Therefore, $A = 10$, and $B = 0$, so $x_p(t) = 10\cos 4t$. The general solution of this DE is then

$$x(t) = x_c(t) + x_p(t) = c_1\cos 5t + c_2 \sin 5t + 10\cos 4t,$$

and it is to this function that we apply the initial conditions. Since

$$x'(t) = -5c_1 \sin 5t + 5c_2\cos 5t - 40\sin 4t,$$

evaluating these equations at $t = 0$ gives the system $c_1 + 10 = 0$ and $5c_2 = 90$. Hence, $c_2 = 18$, and $c_1 = -10$.

Finally, we combine the $\cos 5t$ and $\sin 5t$ terms into a single function $C\cos(5t - \alpha)$. Then $C = \sqrt{(-10)^2 + 18^2} = 2\sqrt{106}$, and $\tan\alpha = c_2/c_1 = 18/(-10) = -9/5$. Furthermore, we must take α so that $\cos\alpha < 0$ to match $c_1 = -10$, so $\alpha = \pi + \tan^{-1}(-9/5) \approx 2.08$. Hence, as the sum of two oscillations,

$$x(t) = 2\sqrt{106}\cos(5t - \pi + \tan^{-1}(9/5)) + 10\cos 4t.$$ ■

3.6.8. Find the steady periodic solution $x_{sp}(t) = C\cos(\omega t - \alpha)$ of the equation $x'' + 3x' + 5x = -4\cos 5t$. Then graph $x_{sp}(t)$ together with the adjusted forcing function $\tilde{F}(t) = F(t)/m\omega^2$.

Solution: We determine $x_{sp}(t)$, first assuming it has the general form $A\cos 5t + B\sin 5t$. Then

$$x'_{sp}(t) = -5A\sin 5t + 5B\cos 5t, \qquad x''_{sp}(t) = -25A\cos 5t - 25B\sin 5t,$$

so plugging this into the DE, we have

$$\begin{aligned} x'' + 3x' + 5x &= -25A\cos 5t - 25B\sin 5t - 15A\sin 5t + 15B\cos 5t + 5A\cos 5t + 5B\sin 5t \\ &= (15B - 20A)\cos 5t + (-15A - 20B)\sin 5t = -4\cos 5t. \end{aligned}$$

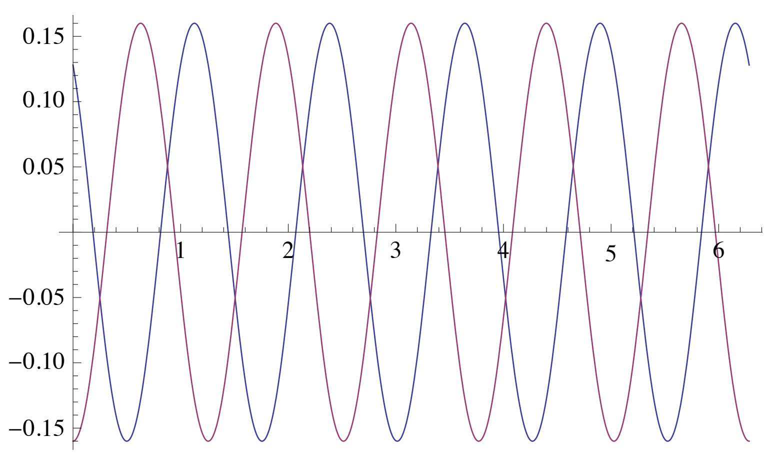
Hence, $-15A - 20B = 0$ and $15B - 20A = -4$, so $B = -\frac{3}{4}A$, and $-\frac{45}{4}A - 20A = -4$, so $A = \frac{16}{125}$. Then $B = -\frac{12}{125}$. Consequently,

$$C = \sqrt{A^2 + B^2} = \frac{\sqrt{12^2 + 16^2}}{125} = \frac{20}{125} = \frac{4}{125}, \quad \tan\alpha = \frac{B}{A} = -\frac{3}{4}.$$

Taking a choice for α in $[0, 2\pi)$, $\alpha = 2\pi - \tan^{-1}(3/4)$, so

$$x_{sp}(t) = \frac{4}{25}\cos(5t - 2\pi + \tan^{-1}(3/4)).$$

Plotting this against $\tilde{F}(t) = -\frac{4}{25}\cos 5t$, we have



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3.6.12. For the differential equation $x'' + 6x' + 13x = 10\sin 5t$, find and plot both the steady periodic solution $x_{sp}(t) = C\cos(\omega t - \alpha)$ and the solution $x(t) = x_{tr}(t) + x_{sp}(t)$ matching the initial conditions $x(0) = 0$ and $x'(0) = 0$.

Solution: We first find $x_{sp}(t)$, in the form $A\cos 5t + B\sin 5t$. From Problem 3.6.8, we reuse the derivatives of this function, so that

$$\begin{aligned} x'' + 6x' + 13x &= -25A\cos 5t - 25B\sin 5t - 30A\sin 5t + 30B\cos 5t + 13A\cos 5t + 13B\sin 5t \\ &= (30B - 12A)\cos 5t + (-30A - 12B)\sin 5t = 10\sin 5t. \end{aligned}$$

Then $30B - 12A = 0$ and $-30A - 12B = 10$, so $B = \frac{2}{5}A$, and then $-30A - \frac{24}{5}A = 10$. Then $A = -\frac{25}{87}$, so $B = -\frac{10}{87}$. Computing C and α,

$$C = \sqrt{A^2 + B^2} = \frac{\sqrt{25^2 + 10^2}}{87} = \frac{5\sqrt{29}}{87} = \frac{5}{3\sqrt{29}}, \quad \alpha = \pi + \tan^{-1}\frac{-10}{-25} = \pi + \tan^{-1}\frac{2}{5}.$$

Then $x_{sp}(t) = \frac{5}{3\sqrt{29}}\cos(5t - \pi - \tan^{-1}\frac{2}{5})$.

Next, we compute the solution $x(t)$ matching the initial conditions. We first find the general form of the transient solution: since the homogeneous equation has the characteristic equation $r^2 + 6r + 13 = 0$, with roots $r = -3 \pm 2i$, the transient solution is of the form

$$x_{tr}(t) = c_1 e^{-3t}\cos 2t + c_2 e^{-3t}\sin 2t,$$

with derivative

$$x_{tr}'(t) = c_1 e^{-3t}(-3\cos 2t - 2\sin 2t) + c_2 e^{-3t}(-3\sin 2t + 2\cos 2t).$$

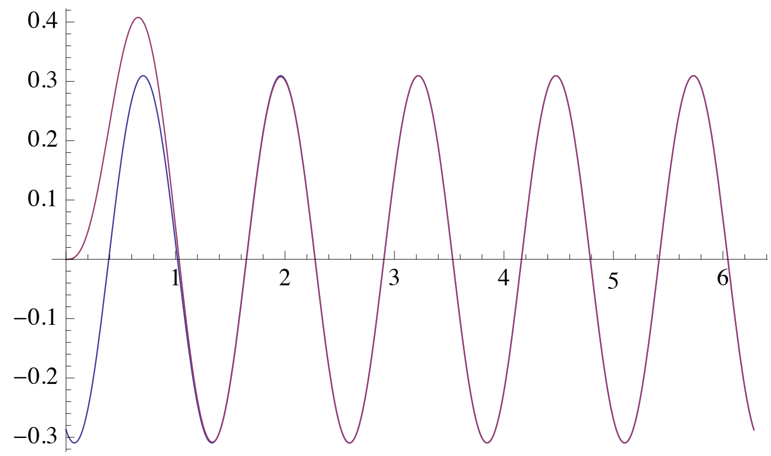
Then $x(t) = x_{sp}(t) + x_{tr}(t)$, so, matching the initial conditions,

$$\begin{aligned} x(0) = x_{sp}(0) + x_{tr}(0) &= -\frac{25}{87} + c_1 = 0, \\ x'(0) = x_{sp}'(0) + x_{tr}'(0) &= -\frac{50}{87} - 3c_1 + 2c_2 = 0. \end{aligned}$$

Then $c_1 = \frac{25}{87}$, and $c_2 = \frac{1}{2}(3c_1 + \frac{50}{87}) = \frac{125}{174}$. Combining the terms in $x_{tr}(t)$ into a single trigonometric function $C_1 e^{-3t}\cos(2t - \beta)$, we then have

$$C = \frac{\sqrt{50^2 + 125^2}}{174} = \frac{25}{6\sqrt{29}}, \quad \beta = \tan^{-1}\frac{125}{50} = \tan^{-1}\frac{5}{2},$$

so $x_{tr}(t) = \frac{25}{6\sqrt{29}}\cos(2t - \tan^{-1}\frac{5}{2})$. We plot $x_{sp}(t)$ and $x(t)$ below:



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3.6.18. Consider the mass-spring-dashpot system $mx'' + cx' + kx = F_0 \cos \omega t$ with $m = 1$, $c = 10$, $k = 650$, and $F_0 = 100$ (in mks units). Find and sketch the amplitude $C(\omega)$ of steady periodic oscillations with frequency ω, and find the practical resonance frequency ω, if it exists.

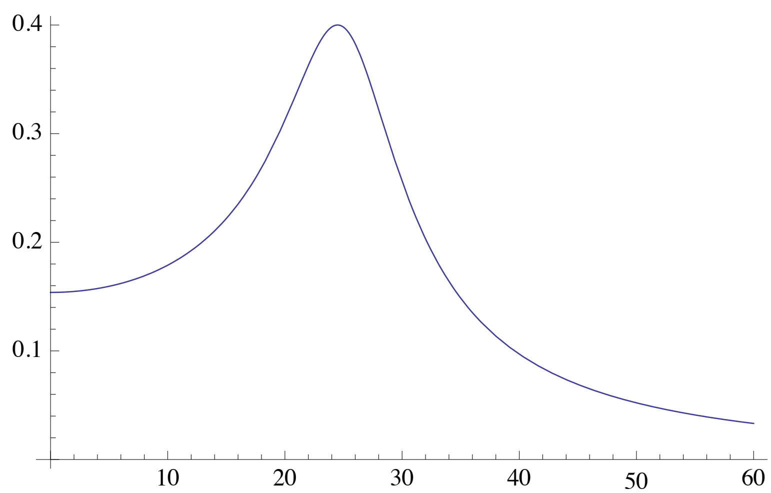
Solution: From the computations in this section, the amplitude is given by

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} = \frac{100}{\sqrt{(650 - \omega^2)^2 + 100\omega^2}} = \frac{100}{\sqrt{422{,}500 - 1200\omega^2 + \omega^4}}.$$

We check whether practical resonance is possible: since $c^2 = 10^2 = 100$ and $2km = 2(1)(650) = 1300$, $c^2 < 2km$, so it is. The frequency maximizing $C(\omega)$ is then

$$\omega_m = \sqrt{\frac{2km - c^2}{2m^2}} = \sqrt{\frac{1300 - 100}{2}} = \sqrt{600} = 10\sqrt{6} \approx 24.5.$$

Below is a plot of $C(\omega)$, which clearly has a maximum at that frequency:



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3.6.28. As indicated by the cart-with-flywheel example discussed in this section, an unbalanced rotating machine part typically results in a force having amplitude proportional to the square of the frequency ω.

(a) Show that the amplitude of the steady periodic solution of the differential equation

$$mx'' + cx' + kx = mA\omega^2 \cos \omega t$$

is given by

$$C(\omega) = \frac{mA\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

(b) Suppose that $c^2 < 2mk$. Show that the maximum amplitude occurs at the frequency ω_m given by

$$\omega_m = \sqrt{\frac{k}{m}\left(\frac{2km}{2km - c^2}\right)}.$$

Thus the resonance frequency in this case is larger than the natural frequency $\omega_0 = \sqrt{k/m}$. (Suggestion: maximize the square of C.)

Solution (a): We note that, although the forcing term is now $mA\omega^2 \cos \omega t$, the amplitude $mA\omega^2$ does not vary with t, and so is a constant under differentiation with respect to t. Then the same formula as in Equation 3.6.21 applies with $F_0 = mA\omega^2$, so the amplitude is

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{mA\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

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Solution (b): As suggested by the hint, we instead maximize $C(\omega)^2 = \frac{A^2m^2\omega^2}{(k-m\omega^2)^2+c^2\omega^2}$. By the quotient rule,

$$\begin{aligned}(C(\omega)^2)' &= \frac{(4A^2m^2\omega^3)((k - m\omega^2)^2 + c^2\omega^2) - (A^2m^2\omega^4)(2(-2m\omega)(k - m\omega^2)^2 + 2c^2\omega)}{((k - m\omega^2) + c^2\omega^2)^2} \\ &= \frac{2A^2m^2\omega^3(2((k - m\omega^2)^2 + c^2\omega^2) - (2m^2\omega^4 - 2km\omega^2 + c^2\omega^2))}{((k - m\omega^2)^2 + c^2\omega^2)^2}\end{aligned}$$

Setting the numerator equal to 0, we have that either $\omega = 0$ or

$$\begin{aligned}0 &= 2((k - m\omega^2)^2 + c^2\omega^2) - (2m^2\omega^4 - 2km\omega^2 + c^2\omega^2) \\ &= 2m^2\omega^4 - 4km\omega^2 + 2k^2 + 2c^2\omega^2 - 2m^2\omega^4 + 2km\omega^2 - c^2\omega^2 \\ &= (c^2 - 2km)\omega^2 + 2k^2.\end{aligned}$$

Therefore, $(2km - c^2)\omega^2 = 2k^2$, so

$$\omega = \sqrt{\frac{2k^2}{2km - c^2}} = \sqrt{\frac{k}{m}\left(\frac{2km}{2km - c^2}\right)}.$$

A routine verification shows that $C(\omega)$ is indeed a maximum here, if $c^2 < 2km$ so that the square root exists. ∎

4.1.2. Transform the differential equation $x^{(4)} + 6x'' - 3x' + x = \cos 3t$ into an equivalent system of first-order differential equations.

Solution: Since we have a fourth derivative of x in the system, we introduce 4 variables: $x_1 = x$, $x_2 = x_1' = x'$, $x_3 = x_2'$, and $x_4 = x_3'$. Then the original DE becomes $x_4' + 6x_3 - 3x_2 + x_1 = \cos 3t$, so we obtain the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= -x_1 + 3x_2 - 6x_3 + \cos 3t \end{aligned}$$

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4.1.8. Transform the system of differential equations $x'' + 3x' + 4x - 2y = 0$, $y'' + 2y' - 3x + y = \cos t$ into an equivalent system of first-order differential equations.

Solution: We introduce the following variables to rewrite the system: $x_1 = x$, $x_2 = x_1 = x'$, $y_1 = y$, and $y_2 = y_1'$. Then $x_2' + 3x_2 + 4x_1 - 2y_1 = 0$ and $y_2' + 2y_2 - 3x_1 + y_1 = \cos t$, so we have the following linear system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -4x_1 - 3x_2 + 2y_1 \\ y_1' &= y_2 \\ y_2' &= 3x_1 - y_1 - 2y_2 + \cos t \end{aligned}$$

∎

4.1.24. Derive the equations

$$m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2, \qquad m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2$$

from the displacements from equilibrium of the two masses in Figure 4.1.11.

Solution: At a given time t, the net displacements of these three springs are x_1, $x_2 - x_1$, and $-x_2$, respectively, so the corresponding forces are $F_1 = -k_1 x_1$, $F_2 = -k_2(x_2 - x_1) = k_2 x_1 - k_2 x_2$, and $F_3 = -k_3(-x_2) = k_3 x_2$. The net forces acting on the masses m_1 and m_2 are then $F_1 - F_2$ and $F_2 - F_3$, so Newton's law provides the equations

$$\begin{aligned} m_1 x_1'' &= F_1 - F_2 = -k_1 x_1 - k_2 x_1 + k_2 x_2 = -(k_1 + k_2)x_1 + k_2 x_2, \\ m_2 x_2'' &= F_2 - F_3 = k_2 x_1 - k_2 x_2 - k_3 x_2 = k_2 x_1 - (k_2 + k_3)x_2, \end{aligned}$$

as desired. ∎