Homework #10 Solutions

Problems

- Section 5.1: 6, 12, 14, 24, 26, 36.
- Section 5.2: 4, 16, 18. On #4 and #16, make only a rough sketch of some solution curves, including ones along the eigenvector directions.

5.1.6. Let

$$A_1 = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}.$$

(a) Show that $A_1B = A_2B$ and note that $A_1 \neq A_2$. Thus, the cancellation law does not hold for matrices; that is, if $A_1B = A_2B$ and $B \neq 0$, it does not follow that $A_1 = A_2$.

(b) Let $A = A_1 - A_2$ and use part (a) to show that $AB = 0$. Thus, the product of two nonzero matrices may be the zero matrix.

Solution (a): We compute:

$$A_1B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2(2)+1(1) & 2(4)+1(2) \\ -3(2)+2(1) & -3(4)+2(2) \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ -4 & -8 \end{bmatrix}$$

$$A_2B = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1(2)+3(1) & 1(4)+3(2) \\ -1(2)+-2(1) & -1(4)-2(2) \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ -4 & -8 \end{bmatrix}$$

These products A_1B and A_2B are then the same matrix. On the other hand,

$$A = A_1 - A_2 = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

which is not the 0 matrix. ∎

Solution (b): Since $A_1B = A_2B$, $A_1B - A_2B = 0$, so by the distributivity of matrix multiplication, $(A_1 - A_2)B = 0$. Defining $A = A_1 - A_2$, $AB = 0$. ∎

5.1.12. Write the system $x' = 3x - 2y$, $y' = 2x + y$ in the form $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{f}(t)$.

Solution: Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3x - 2y \\ 2x + y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Taking

$$P(t) = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

this system is $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{f}(t)$. ■

5.1.14. Write the system $x' = tx - e^t y + \cos t$, $y' = e^{-t}x + t^2 y - \sin t$ in the form $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{f}(t)$.

Solution: Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} tx - e^t y + \cos t \\ e^{-t}x + t^2 y - \sin t \end{bmatrix} = \begin{bmatrix} t & e^t \\ e^{-t} & t^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}.$$

Taking

$$P(t) = \begin{bmatrix} t & e^t \\ e^{-t} & t^2 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix},$$

this system is $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{f}(t)$. ■

5.1.24. First, verify that $\mathbf{x}_1 = e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = e^{2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ are solutions to the system $\mathbf{x}' = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \mathbf{x}$. Then use the Wronkian to show that they are linearly independent. Finally, write the general solution of the system.

Solution: We first check that these are solutions, letting A denote the coefficient matrix in the system, by evaluating $\mathbf{x}'$ and $A\mathbf{x}$ separately and checking that they are equal:

$$\begin{aligned}
\mathbf{x}_1' &= 3e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{3t} \begin{bmatrix} 3 \\ -3 \end{bmatrix} \\
\mathbf{x}_2' &= 2e^{2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = e^{3t} \begin{bmatrix} 2 \\ -4 \end{bmatrix} \\
A\mathbf{x}_1 &= e^{3t} \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = e^{3t} \begin{bmatrix} 4(1) + 1(-1) \\ -2(1) + 1(-1) \end{bmatrix} = e^{3t} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \mathbf{x}_1' \\
A\mathbf{x}_2 &= e^{2t} \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = e^{2t} \begin{bmatrix} 4(1) + 1(-2) \\ -2(1) + 1(-2) \end{bmatrix} = e^{2t} \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \mathbf{x}_2'
\end{aligned}$$

We then compute their Wronskian $W(\mathbf{x}_1, \mathbf{x}_2)$:

$$W(\mathbf{x}_1, \mathbf{x}_2)(t) = \det \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{bmatrix} = \begin{vmatrix} e^{3t} & e^{2t} \\ -e^{3t} & -2e^{2t} \end{vmatrix} = e^{3t}(-2e^{2t}) - (-e^{3t})e^{2t} = -e^{5t}.$$

Since this function is not identically 0 (and in fact is never 0), the solutions are linearly independent. The general solution is then

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ -2e^{3t} \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} + c_2 e^{2t} \\ -c_1 e^{3t} - 2c_2 e^{2t} \end{bmatrix}. \qquad \blacksquare$$

5.1.26. First, verify that $\mathbf{x}_1 = e^t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = e^{3t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{x}_3 = e^{5t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ are solutions to the system $\mathbf{x}' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \mathbf{x}$. Then use the Wronkian to show that they are linearly independent. Finally, write the general solution of the system.

Solution: We first check that these are solutions, letting A denote the coefficient matrix in the system, by evaluating $\mathbf{x}'$ and $A\mathbf{x}$ separately and checking that they are equal. The derivatives are:

$$\mathbf{x}_1' = e^t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}_2' = 3e^{3t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = e^{3t} \begin{bmatrix} -6 \\ 0 \\ 3 \end{bmatrix} \qquad \mathbf{x}_3' = 5e^{5t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = e^{5t} \begin{bmatrix} 10 \\ -10 \\ 5 \end{bmatrix}$$

Then the matrix-vector products are

$$A\mathbf{x}_1' = e^t \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 3(2) - 2(2) + 0(1) \\ -1(2) + 3(2) - 2(1) \\ 0(2) + 1(2) + 3(1) \end{bmatrix} = e^t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \mathbf{x}_1'$$

$$A\mathbf{x}_2' = e^t \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 3(-2) - 2(0) + 0(1) \\ -1(-2) + 3(0) - 2(1) \\ 0(-2) + 1(0) + 3(1) \end{bmatrix} = e^t \begin{bmatrix} -6 \\ 0 \\ 3 \end{bmatrix} = \mathbf{x}_2'$$

$$A\mathbf{x}_3' = e^t \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 3(2) - 2(-2) + 0(1) \\ -1(2) + 3(-2) - 2(1) \\ 0(2) + 1(-2) + 3(1) \end{bmatrix} = e^t \begin{bmatrix} 10 \\ -10 \\ 5 \end{bmatrix} = \mathbf{x}_3'$$

The Wronskian of these three solutions is then

$$W(t) = \begin{vmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t) \end{vmatrix} = \begin{vmatrix} 2e^t & -2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ 1e^t & 1e^{3t} & e^{5t} \end{vmatrix}$$

$$= e^t e^{3t} e^{5t} \begin{vmatrix} 2 & -2 & 2 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{vmatrix} = e^{9t} \left(-2 \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} \right) = 16e^{9t},$$

so the solutions are linearly independent. Thus, the general solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t) = \begin{bmatrix} 2c_1 e^t - 2c_2 e^{3t} + 2c_3 e^{5t} \\ 2c_1 e^t - 2c_3 e^{5t} \\ c_1 e^t + c_2 e^{3t} + c_3 e^{5t} \end{bmatrix}. \qquad \blacksquare$$

5.1.36. Given that $\mathbf{x}_1 = e^{2t}\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{x}_2 = e^{-t}\begin{bmatrix}1\\0\\-1\end{bmatrix}$, and $\mathbf{x}_1 = e^{-t}\begin{bmatrix}0\\1\\-1\end{bmatrix}$ are linearly independent solutions to the system $\mathbf{x}' = \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\mathbf{x}$, find the solution matching the initial conditions $x_1(0) = 10$, $x_2(0) = 12$, $x_3(0) = -1$.

Solution: Letting $\mathbf{b} = \begin{bmatrix}10\\12\\-1\end{bmatrix}$, we then wish to solve the vector equation

$$c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) + c_3\mathbf{x}_3(0) = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3\begin{bmatrix}0\\1\\-1\end{bmatrix} = \mathbf{b}.$$

We can rewrite this linear system in the c_i as a matrix-vector equation $A\mathbf{c} = \mathbf{b}$, with the columns of A coming from the column vectors $\mathbf{x}_i(0)$:

$$\begin{bmatrix}1&1&0\\1&0&1\\1&-1&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1&1&0\\1&0&1\\1&-1&-1\end{bmatrix}\mathbf{c} = \mathbf{b}.$$

Then row reduction of the augmented matrix $[A|\mathbf{b}]$ to $[I|\mathbf{c}]$ will produce the solution $\mathbf{c}$:

$$\left[\begin{array}{ccc|c}1&1&0&10\\1&0&1&12\\1&-1&-1&-1\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&0&10\\0&-1&1&2\\0&-2&-1&-11\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&0&10\\0&1&-1&-2\\0&-2&-1&-11\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1&1&0&10\\0&1&-1&-2\\0&0&-3&-15\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&0&10\\0&1&-1&-2\\0&0&1&5\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}1&1&0&10\\0&1&0&3\\0&0&1&5\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&0&7\\0&1&0&3\\0&0&1&5\end{array}\right]$$

Then $c_1 = 7$, $c_2 = 3$, $c_3 = 5$, so the solution is

$$\mathbf{x}(t) = 7\mathbf{x}_1(t) + 3\mathbf{x}_2(t) + 5\mathbf{x}_3(t)$$

■

5.2.4. Apply the eigenvalue method to find a general solution to the system $x_1' = 4x_1 + x_2$, $x_2' = 6x_1 - x_2$. Sketch some solution curves, including ones along the eigenvector directions.

Solution: Writing $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the system is $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}$. We compute the eigenvalues and eigenvectors of A. First, $\det(A - \lambda I)$ is

$$\begin{vmatrix} 4-\lambda & 1 \\ 6 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 6 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2).$$

The roots of this polynomial are the eigenvalues, which we enumerate $\lambda_1 = 5$ and $\lambda_2 = -2$. We compute the eigenvectors associated to these eigenvalues from the solutions $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ to $(A - \lambda I)\mathbf{v} = \mathbf{0}$. For $\lambda_1 - 5$, this is the linear system

$$\begin{bmatrix} -1 & 1 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then row reduction of $A - 5I$ yields

$$\begin{bmatrix} -1 & 1 \\ 6 & -6 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix},$$

representing the only nontrivial equation $-a + b = 0$. Then $a = b$, so taking $a = 1$, the only eigenvector for $\lambda_1 = 5$ (up to scalar multiples) is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Repeating this process for $\lambda_2 = -2$, we row reduce

$$A + 2I = \begin{bmatrix} 6 & 1 \\ 6 & 1 \end{bmatrix} \sim \begin{bmatrix} 6 & 1 \\ 0 & 0 \end{bmatrix},$$

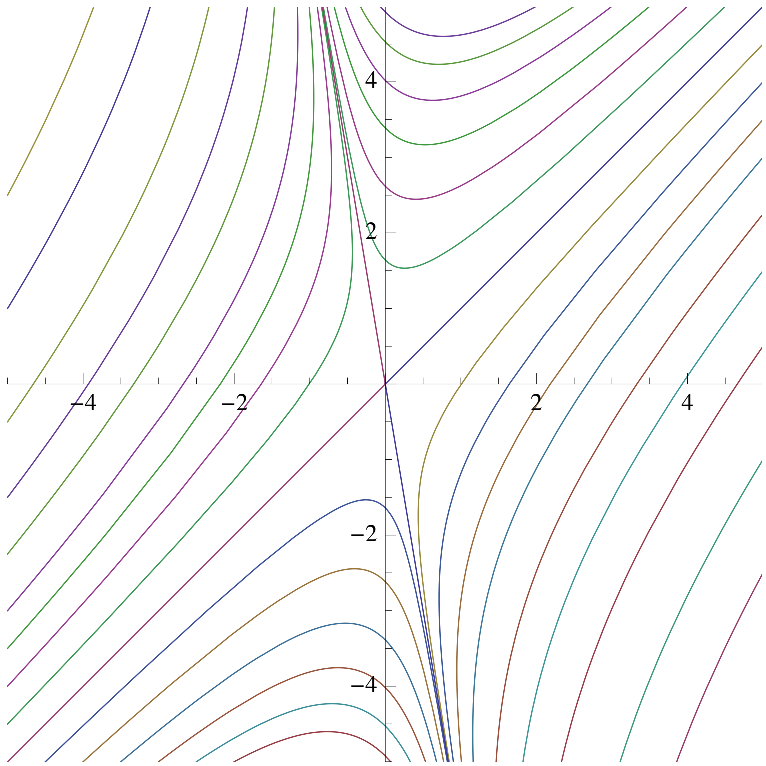
so $6a + b = 0$, and $b = -6a$. Taking $a = 1$, $b = 6$, so the only eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$. Therefore, two linearly independent solutions are

$$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1 = e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2 = e^{-2t} \begin{bmatrix} 1 \\ -6 \end{bmatrix},$$

so the general solution is

$$\mathbf{x}(t) = c_1 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -6 \end{bmatrix}.$$

We plot the solutions in the x_1x_2-plane for different values of c_1 and c_2, including along the eigenvector directions:



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5.2.16. Apply the eigenvalue method to find a general solution to the system $x_1' = -50x_1 + 20x_2$, $x_2' = 100x_1 - 60x_2$. Sketch some solution curves, including ones along the eigenvector directions.

Solution: Writing $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the system is $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} -50 & 20 \\ 100 & -60 \end{bmatrix}$. We compute the eigenvalues and eigenvectors of A. First, $\det(A - \lambda I)$ is

$$\begin{vmatrix} -50-\lambda & 20 \\ 100 & -60-\lambda \end{vmatrix} = (-50-\lambda)(-60-\lambda) - 2000 = \lambda^2 + 110\lambda + 1000.$$

This factors as $(\lambda + 10)(\lambda + 100)$, so the eigenvalues are $\lambda_1 = -10$ and $\lambda_2 = -100$. We compute eigenvectors $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ for each eigenvalue. First, we row reduce $A - \lambda_1 I$:

$$A + 10I = \begin{bmatrix} -40 & 20 \\ 100 & -50 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix},$$

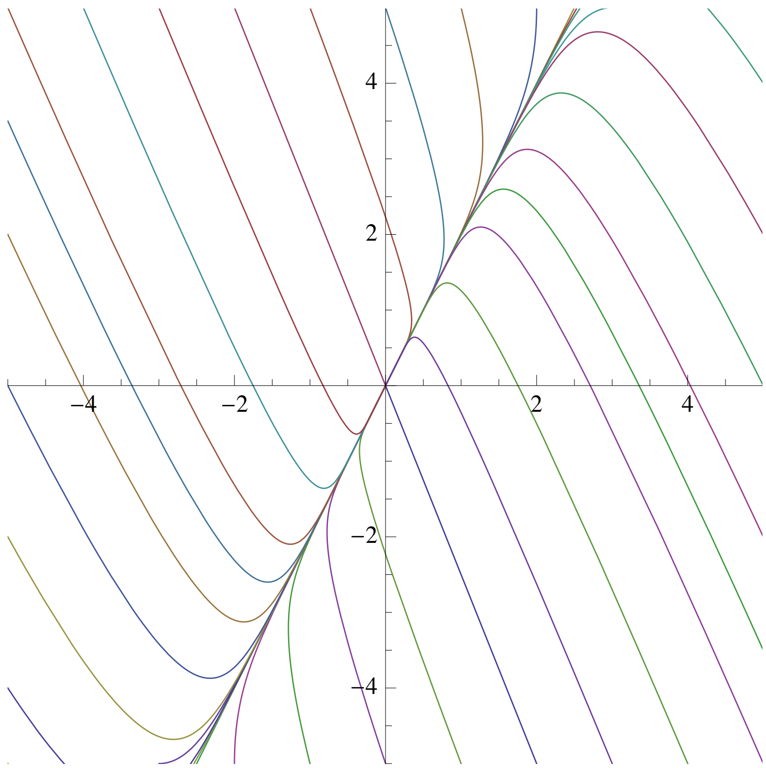
so $-2a + b = 0$, and $b = 2a$. Taking $a = 1$, $b = 2$, so an eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Next, for $\lambda_2 = -100$,

$$A + 100I = \begin{vmatrix} 50 & 20 \\ 100 & 40 \end{vmatrix} \sim \begin{vmatrix} 5 & 2 \\ 0 & 0 \end{vmatrix},$$

so $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ is an eigenvector for λ_2. Hence, the general solution is

$$\mathbf{x}(t) = c_1 e^{-10t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-100t} \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

We plot the solutions in the $x_1 x_2$-plane for different values of c_1 and c_2, including along the eigenvector directions:



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5.2.18. Apply the eigenvalue method to find a general solution to the system $x_1' = x_1 + 2x_2 + 2x_3$, $x_2' = 2x_1 + 7x_2 + x_3$, $x_3' = 2x_1 + x_2 + 7x_3$.

Solution: Writing $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, the system is $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix}$. We compute the eigenvalues and eigenvectors of A. First,

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 7-\lambda & 1 \\ 2 & 1 & 7-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} 7-\lambda & 1 \\ 1 & 7-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 2 & 7-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 7-\lambda \\ 2 & 1 \end{vmatrix} \\ &= (1-\lambda)((7-\lambda)^2-1) - 2(14-2\lambda-2) + 2(2-14+2\lambda) \\ &= (1-\lambda)(\lambda^2-14\lambda+48) + (4\lambda-24) + (4\lambda-24) \\ &= -\lambda^3+\lambda^2+14\lambda^2-14\lambda-48\lambda+48+8\lambda-48 \\ &= -\lambda^3+15\lambda^2-54\lambda = -\lambda(\lambda-9)(\lambda-6).\end{aligned}$$

Therefore, we obtain 3 distinct eigenvalues, $\lambda_1 = 0$, $\lambda_2 = 6$, and $\lambda_3 = 9$. We compute eigenvectors $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for each of them. First, for λ_1, we row reduce $A - 0I = A$:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $a + 4c = 0$ and $b - c = 0$, so $a = -4c$ and $b = c$, but c is free. Taking $c = 1$, we have an eigenvector $\mathbf{v}_1 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$ for $\lambda_1 = 0$. For $\lambda_2 = 6$,

$$A - 6I = \begin{bmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -9 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $a = 0$ and $b + c = 0$, so $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is a reasonable choice for an eigenvector for λ_2.

Finally, for $\lambda_3 = 9$,

$$A - 9I = \begin{bmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} -4 & 1 & 1 \\ 2 & -2 & 1 \\ 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} -4 & 0 & -2 \\ 2 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $2a - c = 0$ and $b - c = 0$, so $b = c = 2a$. Then we may take $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ as an eigenvector for $\lambda_3 = 9$. Hence, the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 e^{9t} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

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