Homework #12 Solutions

Problems

- Section 5.5: 2, 4, 12, 22, 28
- Section 5.6: 2, 8, 24

5.5.2. Find a fundamental matrix for the system $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \mathbf{x}$, and apply $\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}_0$ to find a solution matching the initial condition $\mathbf{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Solution: We first find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$. To find the eigenvalues, we solve $\det(A - \lambda I) = 0$ for λ:

$$\det(A - \lambda I) = \begin{bmatrix} 2-\lambda & -1 \\ -4 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda = \lambda(\lambda - 4) = 0.$$

Then $\lambda_1 = 0$ and $\lambda_2 = 4$ are the distinct, real eigenvalues of A. To find eigenvectors, we row-reduce $A - \lambda I$ for each λ_i:

$$A - 0I = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \qquad A - 4I = \begin{bmatrix} -2 & -1 \\ -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

Then we may take eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, so we build a fundamental matrix from the solutions $e^{\lambda_i t}\mathbf{v}_i$:

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix}$$

Then $\Phi(0) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$, so $\Phi(0)^{-1} = \frac{1}{-4}\begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$. Then, using $\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}(0)$,

$$\mathbf{x}(t) = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ \frac{5}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} + \frac{5}{4}e^{4t} \\ \frac{3}{2} - \frac{5}{2}e^{4t} \end{bmatrix}.$$

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5.5.4. Find a fundamental matrix for the system $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x}$, and apply $\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}_0$ to find a solution matching the initial condition $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution: We first find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$. To find the eigenvalues, we solve $\det(A - \lambda I) = 0$ for λ:

$$\det(A - \lambda I) = \begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0.$$

Then $\lambda_1 = 2$ is the only eigenvalue of A, with multiplicity. To find an eigenvector, we row-reduce $A - 2I$:

$$A - 2I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Then there is only one linearly independent eigenvector for $\lambda = 2$, which we may take to be $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. To build two linearly independent solutions to the system of DEs, we find a generalized eigenvector $\mathbf{v}_2$ so that $(A - 2I)\mathbf{v}_2 = \mathbf{v}_1$. Row-reducing $[A - 2I \mid \mathbf{v}_1]$, this is

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Then we may take $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so two linearly independent solution to the system are $e^{2t}\mathbf{v}_1$ and $e^{2t}(t\mathbf{v}_1 + \mathbf{v}_2)$. We arrange these into a fundamental matrix:

$$\Phi(t) = \begin{bmatrix} e^{2t}\mathbf{v}_1 & e^{2t}(t\mathbf{v}_1 + \mathbf{v}_2) \end{bmatrix} = e^{2t} \begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix}.$$

Then $\Phi(0) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, so $\Phi(0)^{-1} = \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. Then, using $\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}(0)$,

$$\mathbf{x}(t) = e^{2t} \begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{2t} \begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 1+t \\ t \end{bmatrix}.$$

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5.5.12. Compute the matrix exponential e^{At} for the system $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$.

Solution: Regrettably, A is not of the form where we may write down e^{At} fairly directly, so we instead compute it as $\Phi(t)\Phi(0)^{-1}$ for some fundamental matrix $\Phi(t)$. Finding the eigenvalues with $\det(A - \lambda I) = 0$,

$$\begin{vmatrix} 5-\lambda & -4 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0$$

Then $\lambda_1 = 2$ and $\lambda_2 = 1$ are the eigenvalues of A, and from

$$A - 2I = \begin{bmatrix} 3 & -4 \\ 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 \\ 0 & 0 \end{bmatrix} A - I = \begin{bmatrix} 4 & -4 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

we acquire eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then a fundamental matrix $\Phi(t)$ is

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 4e^{2t} & e^t \\ 3e^{2t} & e^t \end{bmatrix}$$

Then $\Phi(0) = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$, so $\Phi(0)^{-1} = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$, and

$$e^{At} = \begin{bmatrix} 4e^{2t} & e^t \\ 3e^{2t} & e^t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 4e^{2t} - 3e^t & -4e^{2t} + 4e^t \\ 3e^{2t} - 3e^t & -3e^{2t} + 4e^t \end{bmatrix}.$$

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5.5.22. Show that the matrix $A = \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix}$ is nilpotent, and use this to compute the matrix exponential e^{At}.

Solution: We compute powers of A:

$$A^2 = AA = \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the power series $e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n$ stops after the $n = 1$ term, so

$$e^{At} = I + At = \begin{bmatrix} 1+6t & 4t \\ -9t & 1-6t \end{bmatrix}.$$

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5.5.28. Use that $A = \begin{bmatrix} 5 & 0 & 0 \\ 10 & 5 & 0 \\ 20 & 30 & 5 \end{bmatrix}$ is the sum of a nilpotent matrix and a multiple of the identity matrix to solve the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$.

Solution: Let $B = \begin{bmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 20 & 30 & 0 \end{bmatrix}$, so that $A = 5I + B$. We show B is nilpotent:

$$B^2 = BB = \begin{bmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 20 & 30 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 20 & 30 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 300 & 0 & 0 \end{bmatrix},$$

so by the placement of the 0s it is clear that $B^3 = B^2 B = 0$. Then

$$e^{Bt} = I + Bt + \frac{1}{2}B^2t^2 = \begin{bmatrix} 1 & 0 & 0 \\ 10t & 1 & 0 \\ 150t^2 + 20t & 30 & 1 \end{bmatrix}$$

$$e^{At} = e^{7It}e^{Bt} = e^{7t} \begin{bmatrix} 1 & 0 & 0 \\ 10t & 1 & 0 \\ 150t^2 + 20t & 30 & 1 \end{bmatrix}$$

Then the solution to the IVP is $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$:

$$\mathbf{x}(t) = e^{7t} \begin{bmatrix} 1 & 0 & 0 \\ 10t & 1 & 0 \\ 150t^2 + 20t & 30 & 1 \end{bmatrix} \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix} = e^{7t} \begin{bmatrix} 40 \\ 400t + 50 \\ 6000t^2 + 2300t + 60 \end{bmatrix}$$

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5.6.2. Use the method of undetermined coefficients to find a particular solution to the system $x' = 2x + 3y + 5$, $y' = 2x + y - 2t$.

Solution: We write this system as $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$, where $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} 5 \\ -2t \end{bmatrix}$. To make sure there are no overlaps with $\mathbf{f}(t)$, we compute the eigenvalues of A:

$$\det A - \lambda I = \begin{vmatrix} 2 - \lambda & 3 \\ 2 & 1 - \lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4)$$

Then the eigenvalues are $\lambda = -1$ and $\lambda = 4$. Since the eigenvalue associated to the functions 5 and $-2t$ in the forcing function is 0, there is no overlap, and we guess $\mathbf{x}_p = \mathbf{a} + t\mathbf{b}$ as a particular solution to the system. Before we plug this guess into the system, we write it as $\mathbf{x}' - A\mathbf{x} = \mathbf{f}(t)$, so that the $\mathbf{x}$-terms are all collected together. Then $\mathbf{x}_p' = \mathbf{b}$, so, plugging this in and separating the constant and the t terms,

$$\begin{aligned}\mathbf{x}' - A\mathbf{x} &= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} 2a_1 + 3a_2 \\ 2a_1 + a_2 \end{bmatrix} - t\begin{bmatrix} 2b_1 + 3b_2 \\ 2b_1 + b_2 \end{bmatrix} \\ &= \begin{bmatrix} -2a_1 - 3a_2 + b_1 \\ -2a_1 - a_2 + b_2 \end{bmatrix} + t\begin{bmatrix} -2b_1 - 3b_2 \\ -2b_1 - b_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -2 \end{bmatrix}\end{aligned}$$

Fortunately, the t-terms involve only the variables b_1 and b_2 from the vector $\mathbf{b}$, so we may solve for them independently of the $\mathbf{a}$-vector. Multiplying the equations by -1, we have $2b_1 + 3b_2 = 0$ and $2b_1 + b_2 = 2$, which we solve with augmented matrix reduction:

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -1 \end{array}\right]$$

So $b_1 = \frac{3}{2}$, and $b_2 = -1$. Plugging these solutions into the constant-vector equations, $-2a_1 - 3a_2 + \frac{3}{2} = 5$ and $-2a_1 - a_2 - 1 = 0$, which we solve:

$$\left[\begin{array}{cc|c} 2 & 3 & -\frac{7}{2} \\ 2 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 3 & -\frac{7}{2} \\ 0 & -2 & \frac{5}{2} \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{5}{4} \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{8} \\ 0 & 1 & -\frac{5}{4} \end{array}\right]$$

Then $a_1 = \frac{1}{8}$ and $a_2 = -\frac{5}{4}$, so $\mathbf{x}_p = \begin{bmatrix} \frac{1}{8} + \frac{3}{2}t \\ -\frac{5}{4} - t \end{bmatrix}$. ■

5.6.8. Use the method of undetermined coefficients to find a particular solution to the system $x' = x - 5y + 2\sin t$, $y' = x - y - 3\cos t$.

Solution: We write this system as $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$, where $A = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$ and $\mathbf{f}(t) = \begin{bmatrix} 2\sin t \\ -3\cos t \end{bmatrix}$. To make sure there are no overlaps with $\mathbf{f}(t)$, we compute the eigenvalues of A:

$$\det A - \lambda I = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 4.$$

Then the eigenvalues are the pure imaginary pair $\lambda = \pm 2i$. Since the $\sin t$ and $\cos t$ correspond to the imaginary pair $\pm i$, there is no overlap, and we guess a solution of the form $\mathbf{x}_p = \mathbf{a}\cos t + \mathbf{b}\sin t$. Again writing the system as $\mathbf{x}' - A\mathbf{x} = \mathbf{f}(t)$, we plug in this guess and collect the $\sin t$ and $\cos t$ terms:

$$\begin{aligned}\mathbf{x}' - A\mathbf{x} &= -\sin t\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \cos t\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \cos t\begin{bmatrix} a_1 - 5a_2 \\ a_1 - a_2 \end{bmatrix} - \sin t\begin{bmatrix} b_1 - 5b_2 \\ b_1 - b_2 \end{bmatrix} \\ &= \cos t\begin{bmatrix} -a_1 + 5a_2 + b_1 \\ -a_1 + a_2 + b_2 \end{bmatrix} + \sin t\begin{bmatrix} -a_1 - b_1 + 5b_2 \\ -a_2 - b_1 + b_2 \end{bmatrix} = \cos t\begin{bmatrix} 0 \\ -3 \end{bmatrix} + \sin t\begin{bmatrix} 2 \\ 0 \end{bmatrix}.\end{aligned}$$

Separating the $\sin t$ and $\cos t$ components, we obtain a system of 4 equations in a_1, a_2, b_1, and b_2, which we solve with augmented row reduction:

$$\left[\begin{array}{cccc|c} -1 & 0 & -1 & 5 & 2 \\ 0 & -1 & -1 & 1 & 0 \\ -1 & 5 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -5 & -2 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 5 & 2 & -5 & -2 \\ 0 & 1 & 1 & -4 & -5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -5 & -2 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -3 & 0 & -2 \\ 0 & 0 & 0 & -3 & -5 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -5 & -2 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{5}{3} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{5}{3} \end{array}\right]$$

Then $\mathbf{x}_p = \cos t \begin{bmatrix} \frac{17}{3} \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} \frac{2}{3} \\ \frac{5}{3} \end{bmatrix}$. ■

5.6.24. Use the method of variation of parameters to solve the IVP $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ with $A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ t^{-2} \end{bmatrix}$, and initial condition $\mathbf{x}(1) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, using that $e^{At} = \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix}$.

Solution: We first find a particular solution $\mathbf{x}_p(t)$ to the nonhomogeneous system using variation of parameters. By the formula, $\mathbf{x}_p(t) = e^{At} \int e^{-At}\mathbf{f}(t)\,dt$, which we evaluate from the inside out:

$$e^{-At}\mathbf{f}(t) = \begin{bmatrix} 1-3t & t \\ -9t & 1+3t \end{bmatrix} \begin{bmatrix} 0 \\ t^{-2} \end{bmatrix} = \begin{bmatrix} t^{-1} \\ t^{-2}+3t^{-1} \end{bmatrix}$$

$$\int e^{-At}\mathbf{f}(t)\,dt = \int \begin{bmatrix} t^{-1} \\ t^{-2}+3t^{-1} \end{bmatrix} dt = \begin{bmatrix} \ln t \\ -t^{-1}+3\ln t \end{bmatrix}$$

$$e^{At} \int e^{-At}\mathbf{f}(t)\,dt = \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix} \begin{bmatrix} \ln t \\ -t^{-1}+3\ln t \end{bmatrix} = \begin{bmatrix} 1+\ln t \\ 3-t^{-1}+3\ln t \end{bmatrix}$$

We now solve for solution matching the initial condition, using that $\mathbf{x}(t) = e^{At}\mathbf{c} + \mathbf{x}_p(t)$. Then at $t = 1$,

$$\mathbf{x}(1) = \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix}_{t=1} \mathbf{c} + \mathbf{x}_p(1) = \begin{bmatrix} 4 & -1 \\ 9 & -2 \end{bmatrix} \mathbf{c} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then $\begin{bmatrix} 4 & -1 \\ 9 & -2 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, so

$$\mathbf{c} = \begin{bmatrix} 4 & -1 \\ 9 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -9 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Finally, the solution is

$$\begin{aligned}\mathbf{x}(t) = e^{At}\mathbf{c} + \mathbf{x}_p(t) &= \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1+\ln t \\ 3 - t^{-1} + 3\ln t \end{bmatrix} \\ &= \begin{bmatrix} 1+t \\ 2+3t \end{bmatrix} + \begin{bmatrix} 1+\ln t \\ 3 - t^{-1} + 3\ln t \end{bmatrix} = \begin{bmatrix} 2+t+\ln t \\ 5+3t-t^{-1}+3\ln t \end{bmatrix}\end{aligned}$$

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