Homework #13 Solutions

Problems

- Section 6.1: 2, 4, 6, 8
- Section 6.2: 2, 8, 14, 24, 30. Graphing directions:
  * Omit the parts asking for graphical verification via a computer system or graphing calculator.
  * On #14, find eigenvectors for the system and use them to make an approximate sketch of some solution curves around the critical point you find.

6.1.2. Find the critical points of the system $x' = x - y$, $y' = x + 3y - 4$ to match it to one of the phase portraits in Figures 6.1.12 through 6.1.19.

Solution: Setting $x' = 0$ and $y' = 0$, we have the system $x - y = 0$ and $x + 3y - 4 = 0$. Then $x = y$, so $4y - 4 = 0$, and $y = 1$. Hence, the only critical point of the system is $(1, 1)$, so it must correspond to Figure 6.1.16. ■

6.1.4. Find the critical points of the system $x' = 2x - 2y - 4$, $y' = x + 4y + 3$ to match it to one of the phase portraits in Figures 6.1.12 through 6.1.19.

Solution: Setting $x' = 0$ and $y' = 0$, we have the system $2x - 2y - 4 = 0$ and $x + 4y + 3 = 0$. Then $x = -4y - 3$, so $2(-4y - 3) - 2y - 4 = 0$, and $-10y - 10 = 0$. Hence, $y = -1$, so $x = 4 - 3 = 1$, and the only critical point is $(1, -1)$. Thus, this system must correspond to Figure 6.1.13. ■

6.1.6. Find the critical points of the system $x' = 2 - 4x - 15y$, $y' = 4 - x^2$ to match it to one of the phase portraits in Figures 6.1.12 through 6.1.19.

Solution: Setting $x' = 0$ and $y' = 0$, we have the system $2 - 4x - 15y = 0$ and $4 - x^2 = 0$. From the second equation, $x = 2$ or $x = -2$. Choosing the first option, $2 - 8 - 15y = 0$, so $y = -6/15 = -2/5$, and $(2, -2/5)$ is a critical point. Choosing the second option, $2 + 8 - 15y = 0$, so $y = 10/15 = 2/3$, and $(-2, 2/3)$ is the other critical point of the system. Therefore, the system must correspond to Figure 6.1.18. ■

6.1.8. Find the critical points of the system $x' = x - y - x^2 + xy$, $y' = -y - x^2$ to match it to one of the phase portraits in Figures 6.1.12 through 6.1.19.

Solution: Setting $x' = 0$ and $y' = 0$, we have the system $x - y - x^2 + xy = 0$ and $-y - x^2 = 0$. From the second equation, $y = -x^2$, which we substitute into the first equation to obtain $x + x^2 - x^2 - x^3 = 0$. This is $x - x^3 = 0$, which factors as $-x(x-1)(x+1) = 0$. Then $x = 0$, $x = 1$, or $x = -1$. Since $y = -x^2$, these give the three critical points $(0,0)$, $(1,-1)$, and $(-1,-1)$, so this system corresponds to Figure 6.1.17. ∎

6.2.2. Apply Theorem 6.2.1 to the system $x' = 4x - y$, $y' = 2x + y$ to determine the type of the critical point $(0,0)$ and whether it is asymptotically stable, stable, or unstable.

Solution: The coefficient matrix for this system is $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$, for which we determine the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2)$$

Then the eigenvalues are $\lambda = 3$ and $\lambda = 2$, which are both positive, real values. Therefore, there is an unstable (improper) node at $(0,0)$. ∎

6.2.8. Apply Theorem 6.2.1 to the system $x' = x - 3y$, $y' = 6x - 5y$ to determine the type of the critical point $(0,0)$ and whether it is asymptotically stable, stable, or unstable.

Solution: The coefficient matrix for this system is $A = \begin{bmatrix} 1 & -3 \\ 6 & -5 \end{bmatrix}$, for which we determine the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -3 \\ 6 & -5-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 13.$$

Solving for the roots of the characteristic polynomial, $\lambda = \frac{-4 \pm \sqrt{16-52}}{2} = -2 \pm 3i$. Since these eigenvalues are complex with a negative real part, the critical point at $(0,0)$ is an asymptotically stable spiral point. ∎

6.2.14. The system $x' = x + y - 7$, $y' = 3x - y - 5$ has a single critical point (x_0, y_0). Apply Theorem 6.2.2 to classify the type and stability of this critical point. Sketch a phase portrait around the critical point using the eigenvectors associated to the eigenvalues you find.

Solution: We first find the critical point (x_0, y_0): $x + y - 7 = 0$ and $3x - y - 5 = 0$, so $y = 3x - 5$. Then $x + 3x - 5 - 7 = 0$, so $x = 12/4 = 3$, and $y = 4$. Thus, $(x_0, y_0) = (3, 4)$ is the only critical point.

Computing the Jacobian matrix of the system, we obtain the constant matrix $J(x, y) = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$. To compute the eigenvalues of $J(x_0, y_0)$, we have

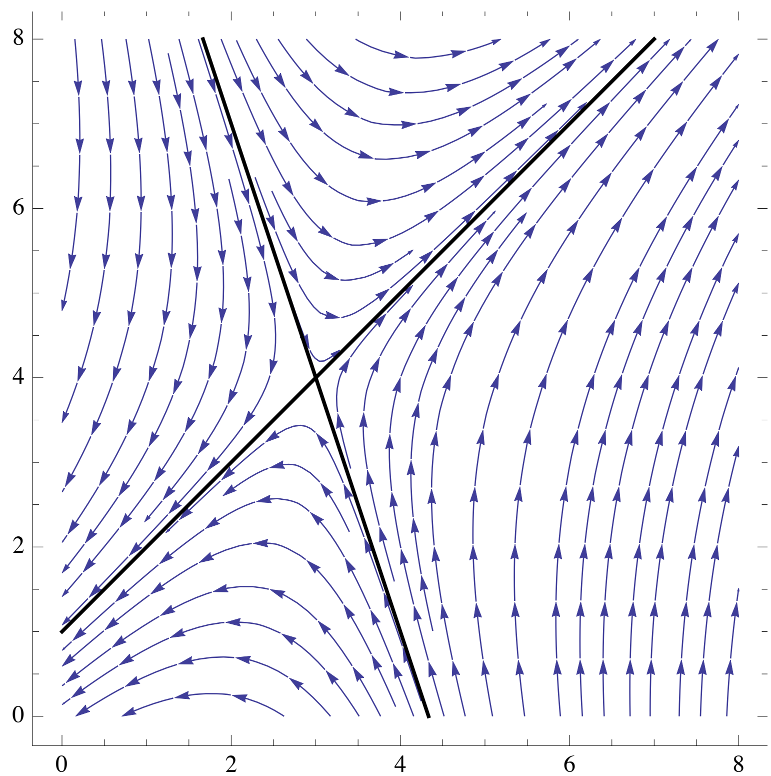
$$\det(J(x_0, y_0) - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2).$$

Therefore, the eigenvalues of $J(x_0, y_0)$ are $\lambda_1 = 2$ and $\lambda_2 = -2$; since they are real and of opposite signs, the system has a saddle point at $(3, 4)$ (which is always unstable).

We also determine the eigenvectors associated to these eigenvalues. Row reducing $J - \lambda_i I$ in these cases,

$$J - 2I = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \qquad J + 2I = \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix},$$

so we take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ as eigenvectors for λ_1 and λ_2. Hence, the straight-line trajectories into $(3, 4)$ run parallel to these directions, as we see on this phase portrait:



■

6.2.24. Investigate the type and stability of the critical point $(0,0)$ of the almost linear system $x' = 5x - 3y + y(x^2 + y^2)$, $y' = 5x + y(x^2 + y^2)$. Also, describe the approximate locations and apparent types of any other critical points of the system.

Solution: Computing the Jacobian matrix of this system, we have

$$J(x,y) = \begin{bmatrix} 5 + 2xy & -3 + x^2 + 3y^2 \\ 5 + 2xy & x^2 + 3y^2 \end{bmatrix}$$

Then $J(0,0) = \begin{bmatrix} 5 & -3 \\ 5 & 0 \end{bmatrix}$, and we compute its eigenvalues:

$$\det J(0,0) - \lambda I = \begin{vmatrix} 5 - \lambda & -3 \\ 5 & -\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 15$$

The roots of this characteristic polynomial are $\lambda = \frac{1}{2}(5 \pm \sqrt{25 - 60}) = \frac{5}{2} \pm i\frac{\sqrt{35}}{2}$. Since this is a complex pair with positive real part, the critical point $(0,0)$ is an unstable spiral point.

We also solve for the other critical points of the system. In this case, we note that $x' - y' = -3y$, so setting both $x' = 0$ and $y' = 0$ gives $-3y = 0$, so $y = 0$. Applying this to the y' equation, $5x = 0$, so $x = 0$ as well. Therefore, $(0,0)$ is the only critical point of the system. ■

6.2.30. Find all critical points of the system $x' = y - 1$, $y' = x^2 - y$, and investigate the type and stability of each.

Solution: Setting $x' = 0$ and $y' = 0$, we have the system $y - 1 = 0$ and $x^2 - y = 0$. Then $y = 1$, so $x^2 - 1 = 0$, and either $x = 1$ or $x = -1$. Therefore, the system has two critical points, $(1,1)$ and $(-1,1)$. We write the Jacobian matrix for the system:

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ 2x & -1 \end{bmatrix}$$

We calculate the eigenvalues of $J(1,1)$ and $J(-1,1)$:

$$\det J(1,1) - \lambda I = \begin{vmatrix} -\lambda & 1 \\ 2 & -1 - \lambda \end{vmatrix} = \lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1)$$

$$\det J(-1,1) - \lambda I = \begin{vmatrix} -\lambda & 1 \\ -2 & -1 - \lambda \end{vmatrix} = \lambda^2 + \lambda + 2$$

Then $J(1,1)$ has eigenvalues $\lambda = -2$ and $\lambda = 1$, so the system has a saddle point at $(1,1)$. The roots of $\lambda^2 + \lambda + 2$ are $\lambda = \frac{1}{2}(-1 \pm \sqrt{1 - 8}) = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$, so since these eigenvalues form a complex pair with a negative real part, the system has an asymptotically stable spiral point at $(-1,1)$. ■