# Solutions to Midterm #2 Practice Problems

**1.** Find general solutions to the following DEs:

(a) $y'' - 6y' + 8y = 0$

*Solution*: The characteristic equation for this constant-coefficient, homogeneous DE is $r^2 - 6r + 8 = 0$, which factors as $(r-2)(r-4) = 0$. Therefore, its roots are $r = 2$ and $r = 4$, so the general solution is

$$y = c_1 e^{2x} + c_2 e^{4x}.$$

(b) $y^{(3)} + 2y'' - 4y' - 8y = 0$

*Solution*: The characteristic equation for this constant-coefficient, homogeneous DE is $r^3 + 2r^2 - 4r - 8 = 0$. We see immediately that $r + 2$ is a factor of the polynomial; factoring it out, we have $(r+2)(r^2-4) = (r+2)(r+2)(r-2) = 0$. Therefore, $r = 2$ is a single root, and $r = -2$ is a double root, so the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 x e^{-2x}.$$

(c) $y'' + 8y' + 20y = 0$

*Solution*: The characteristic equation is $r^2 + 8r + 20 = 0$, which has no immediately obvious solutions. Using the quadratic formula,

$$r = \frac{-8 \pm \sqrt{64-80}}{2} = \frac{-8 \pm \sqrt{-16}}{2} = \frac{-8 \pm 4i}{2} = -4 \pm 2i.$$

From this pair of complex roots, the general solution is

$$y = c_1 e^{-4x} \cos 2x + c_2 e^{-4x} \sin 2x.$$

(d) $y^{(4)} = y$

*Solution*: Writing this DE as $y^{(4)} - y = 0$, its chararcteristic equation is $r^4 - 1 = 0$. This factors as $(r-1)(r+1)(r^2+1) = 0$, so it has roots $r = 1$, $r = -1$, and $r = \pm i$. Therefore, the general solution is

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 \sin x + c_4 \cos x.$$

(e) $y^{(6)} + 8y^{(4)} + 16y'' = 0$
*Solution*: This DE has the characteristic equation $r^6 + 8r^4 + 16r^2 = 0$, which factors as $r^2(r^2+4)^2 = 0$. Then $r = 0$ is a double root, and the pair $r = \pm 2i$ is also a double root, so the general solution is

$$y = c_1 + c_2 x + (c_3 + c_4 x)\cos 2x + (c_5 + c_6 x)\sin 2x.$$

**2.** Find solutions to the following IVPs:

(a) $y'' - 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$
*Solution*: The DE has the characteristic equation $r^2 - 3r + 2 = 0$, and so has roots $r = 1$ and $r = 2$. Hence, its general solution is

$$y = c_1 e^x + c_2 e^{2x}.$$

We match this solution and its derivative, $y' = c_1 e^x + 2c_2 e^{2x}$, to the initial conditions $y(0) = 1$, $y'(0) = 0$ at $x = 0$. We then obtain the linear system

$$c_1 + c_2 = 1, \qquad c_1 + 2c_2 = 0.$$

Then $c_1 = -2c_2$; plugging this into the first equation, $-c_2 = 1$, so $c_2 = -1$ and $c_1 = 2$. Therefore, the solution to the IVP is $y = 2e^x - e^{2x}$.

(b) $9y^{(3)} + 12y'' + 4y' = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = \frac{10}{3}$
*Solution*: The DE has the characteristic equation $9r^3 + 12r^2 + 4r = 0$, which factors as $r(3r+2)^2 = 0$. Hence, $r = 0$ is a single root, and $r = -2/3$ is a double root, so the general solution is

$$y = c_1 + c_2 e^{-2x/3} + c_3 x e^{-2x/3}.$$

Taking derivatives,

$$y' = -\frac{2}{3}c_2 e^{-2x/3} + c_3\left(1 - \frac{2}{3}x\right)e^{-2x/3}, \qquad y'' = \frac{4}{9}c_2 e^{-2x/3} + c_3\left(-\frac{4}{3} + \frac{4}{9}x\right)e^{-2x/3}.$$

Evaluating these functions at $x = 0$ and matching the initial conditions, we obtain the linear system

$$c_1 + c_2 = 0, \qquad -\frac{2}{3}c_2 + c_3 = 1, \qquad \frac{4}{9}c_2 - \frac{4}{3}c_3 = \frac{10}{3}.$$

Then $c_3 = 1 + \frac{2}{3}c_2$, so from the third equation, $\frac{4}{9}c_2 - \frac{4}{3} - \frac{8}{9}c_2 = \frac{10}{3}$, and $c_2 = -\frac{14}{3}\frac{9}{4} = -\frac{21}{2}$. Then $c_3 = 1 - \frac{21}{3} = -6$, so $c_1 = -c_2 = \frac{21}{2}$. Hence, the solution to the IVP is

$$y = \frac{21}{2} - \frac{21}{2}e^{-2x/3} - 6xe^{-2x/3}.$$

**3.** For each of the DEs below, find a particular solution to it:

(a) $y'' - 6y' + 8y = 4x + 5$

*Solution*: From Problem 1(a), we see that the roots of the associated homogeneous DE are $r = 2$ and $r = 4$, neither of which corresponds to the polynomial function $4x + 5$. Therefore, we do not need to modify the guess $y = Ax + B$ to avoid colliding with terms in the complementary solution. Plugging this guess for $y$ into the nonhomogeneous DE, we have

$$0 - 6(A) + 8(Ax + B) = 4x + 5,$$

so equating the constant terms and the coefficients on the $x$ terms, we have that $8A = 4$ and $8B - 6A = 5$. Then $A = 1/2$, so $8B = 5 + 3 = 8$, and $B = 1$. Hence, a particular solution is $y = \frac{1}{2}x + 1$.

(b) $y^{(4)} + 4y'' = 12x - 16 - 8e^{2x}$

*Solution*: Writing the characteristic equation for the associated homogeneous DE, we have $r^4 + 4r^2 = r^2(r^2 + 4) = 0$. This equation has a double root at $r = 0$ and a pair of complex roots $r = \pm 2i$, so the complementary solution is

$$y_c = c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x.$$

From $f(x) = 12x - 16 - 8e^{2x}$, we might initially guess a particular solution of the form $y = Ax + B + Ce^{2x}$, but the first two terms overlap with terms in $y_c$. Hence, we multiply only those terms by $x^2$ to avoid the overlap, obtaining

$$y = Ax^3 + Bx^2 + Ce^{2x}$$

as the form of a solution. Then $y'' = 6Ax + 2B + 4Ce^{2x}$ and $y^{(4)} = 16Ce^{2x}$, so the DE becomes

$$16Ce^{2x} + 24Ax + 8B + 16Ce^{2x} = 12x - 16 - 8e^{2x}.$$

Then $32C = -8$, so $C = -\frac{1}{4}$, $24A = 12$, so $A = \frac{1}{2}$, and $8B = -16$, so $B = -2$. Consequently, a particular solution is

$$y = \frac{1}{2}x^3 - 2x^2 - \frac{1}{4}e^{2x}.$$

(c) $y'' + 2y' - 3y = -4xe^{-3x}$

*Solution*: The roots of the characteristic equation for this DE are $r = -3$ and $r = 1$, so the complementary solution is $y_c = c_1 e^x + c_2 e^{-3x}$. From the $xe^{-3x}$ in the $f(x)$ function, we might guess $y = Axe^{-3x} + Be^{-3x}$ as a particular solution, but the lower-order term overlaps with the complementary solution. Hence, we multiply these terms by $x$ to separate them from $y_c$, and the form of our guess is

$$y = Ax^2 e^{-3x} + Bxe^{-3x}.$$

Then its derivatives are

$$y' = A(2x - 3x^2)e^{-3x} + B(1 - 3x)e^{-3x}, \quad y'' = A(9x^2 - 12x + 2)e^{-3x} + B(9x - 6)e^{-3x}.$$

Plugging these into the original nonhomogeneous DE, and observing that the $x^2e^{-3x}$ terms all cancel, we have

$$A(-12x + 2)e^{-3x} + B(9x - 6)e^{-3x} + 4Axe^{-3x} + B(2 - 6x)e^{-3x} - 3Bxe^{-3x} = -4xe^{-3x}.$$

Grouping the coefficients on the $xe^{-3x}$ and $e^{-3x}$ terms separately, we have the linear system

$$-8A = -4, \qquad 2A - 4B = 0.$$

Then $A = 1/2$, so $B = \frac{1}{2}A = \frac{1}{4}$, and a particular solution is

$$y = \frac{1}{2}x^2e^{-3x} + \frac{1}{4}xe^{-3x}.$$

**4.** Consider a mass of 2 kg attached to a spring with spring constant 18 N/m. Find the displacement $x(t)$ with the initial conditions $x(0) = 4$ m, $x'(0) = 9$ m/s, assuming the following damping $c$ is present. If the system exhibits periodic behavior, find its (possibly time-varying) amplitude and period.

(a) $c = 0$

*Solution*: Since $c = 0$, there is no damping is the system. We compute the circular frequency $\omega_0$ of the system to be $\sqrt{k/m} = \sqrt{18/2} = 3$ rad/s, so the general solution for the displacement is of the form

$$x(t) = A\cos 3t + B\sin 3t.$$

Its velocity is then $x'(t) = -3A\sin 3t + 3B\cos 3t$. Matching these functions to the initial conditions $x(0) = 4$ and $x'(0) = 9$ at $t = 0$, we have

$$A = 4, \qquad 3B = 9,$$

so $A = 4$ and $B = 3$, and therefore $x(t) = 4\cos 3t + 3\sin 3t$. The amplitude is given by $C = \sqrt{A^2 + B^2} = \sqrt{9 + 16} = 5$, and the period $T = 2\pi/\omega_0 = 2\pi/3$ s.

(b) $c = 4$

*Solution*: We now have damping with $c = 4$. The characteristic equation for the DE $mx'' + cx' + kx = 0$ is $2r^2 + 4r + 18 = 0$, so we check the discriminant of this quadratic polynomial to determine whether we obtain real or complex roots. Since this is $c^2 - 4km = 16 - 4(2)(18) = 16 - 144 = -128$, which is negative, we will get complex roots, with values

$$r = \frac{-4 \pm \sqrt{-128}}{2(2)} = \frac{-4 \pm 8\sqrt{2}}{4} = -1 \pm 2\sqrt{2}.$$

Therefore, the general solution for the displacement and its velocity is

$$x(t) = Ae^{-t}\cos 2\sqrt{2}t + Be^{-t}\sin 2\sqrt{2}t$$
$$x'(t) = Ae^{-t}(-\cos 2\sqrt{2}t - 2\sqrt{2}\sin 2\sqrt{2}t) + Be^{-t}(-\sin 2\sqrt{2}t + 2\sqrt{2}\cos 2\sqrt{2}t),$$

which we match to the initial conditions $x(0) = 4$ and $x'(0) = 9$ to obtain the system

$$A = 4, \qquad -A + 2\sqrt{2}B = 9.$$

Then $2\sqrt{2}B = 9 + 4 = 13$, so $B = 13/2\sqrt{2}$, and the displacement is

$$x(t) = 4e^{-t}\cos 2\sqrt{2}t + \frac{13}{2\sqrt{2}}e^{-t}\sin 2\sqrt{2}t.$$

We have that $C = \sqrt{A^2+B^2} = \sqrt{16+169/8} = \sqrt{297/8} = \frac{3}{2}\sqrt{\frac{33}{2}}$, so the amplitude of this solution is $\frac{3}{2}\sqrt{\frac{33}{2}}e^{-t}$. Since the pseudofrequency is $\omega_1 = 2\sqrt{2}$, the period is $2\pi/\omega_1 = \pi/\sqrt{2}$.

(c) $c = 12$

*Solution*: We check the discriminant $c^2 - 4km$ again when $c = 12$: now it is $12^2 - 144 = 0$, so we expect a double root at $r = -c/2m = -12/4 = -3$. Hence, the general forms of the displacement and the velocity are

$$x(t) = c_1e^{-3t} + c_2te^{-3t}, \qquad x'(t) = -3c_1e^{-3t} + c_2(1-3t)e^{-3t}.$$

Matching these to the initial conditions $x(0) = 4$ and $x'(0) = 9$, we have the system $c_1 = 4$ and $-3c_1 + c_2 = 9$, so $c_2 = 9 + 12 = 21$. Thus, the displacement is $x(t) = (4 + 21t)e^{-3t}$. There is no trigonometric component to the solution, so there is no amplitude or period.

(d) $c = 20$

*Solution*: For $c = 20$, we now expect an overdamped system with 2 real roots given by

$$r = \frac{-20 \pm \sqrt{20^2 - 144}}{4} = \frac{-20 \pm \sqrt{256}}{4}\frac{-20 \pm 16}{4} = -5 \pm 4,$$

so the roots are $r = -1$ and $r = -9$. Therefore, the general forms of the displacement and the velocity are

$$x(t) = c_1e^{-t} + c_2e^{-9t}, \qquad x'(t) = -c_1e^{-t} - 9c_2e^{-9t}.$$

Matching these to the initial conditions $x(0) = 4$ and $x'(0) = 9$, we have the system $c_1 + c_2 = 4$, $-c_1 - 9c_2 = 9$. Adding these equations, $-8c_2 = 13$, so $c_2 = -13/8$, and $c_1 = 4 - c_2 = 45/8$. Thus, the displacement is

$$x(t) = \frac{45}{8}e^{-t} - \frac{13}{8}e^{-9t},$$

and as in the critically damped case there is no amplitude or period.

**5.** Find a differential equation with the general solution

$$y = (c_1 + c_2 x)e^{3x} + c_3 e^{-2x}\cos(\sqrt{2}x) + c_4 e^{-2x}\sin(\sqrt{2}x).$$

*Solution*: From the form of this general solution, we expect the characteristic equation of the DE to have a double root at $r = 3$ and a pair of complex roots $r = -2 \pm \sqrt{2}i$. Then the polynomial in the characteristic equation is

$$\begin{aligned}(r-3)^2(r+2-\sqrt{2}i)(r+2+\sqrt{2}i) &= (r^2-6r+9)(r^2+4r+6)\\ &= r^4 + (-6+4)r^3 + (6-24+9)r^2 + (36-36)r + 54\\ &= r^4 - 2r^3 - 9r^2 + 54.\end{aligned}$$

Thus, the differential equation $y^{(4)} - 2y^{(3)} - 9y'' + 54y = 0$ (or any constant multiple of it) corresponds to this characteristic equation, and hence has this general solution.

**6.**

(a) Show that the functions $y_1 = e^{-x}$ and $y_2 = \sin x$ are linearly independent.

*Solution*: We compute the Wronskian $W(x)$ of these two functions:

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & \sin x \\ -e^{-x} & \cos x \end{vmatrix} = e^{-x}\cos x + e^{-x}\sin x = e^{-x}(\cos x + \sin x).$$

Since $W(x)$ is not identically 0 on the real line (as, for example, $W(0) = e^0(1+0) = 1$), these functions are linearly independent.

(b) Show that the functions $y_1 = 1 + \tan^2 x, y_2 = 3 - 2\tan^2 x$, and $y_3 = \sec^2 x$ are not linearly independent.

*Solution*: The easy way to show these function are linearly independent is to note that $1 + \tan^2 x = \sec^2 x$, so $y_1 = y_3$. Thus, $-y_1 + y_3$ is a nontrivial linear combination of these functions equal to 0.

If we do not notice this fact, we proceed by computing the derivatives of $y_1$, $y_2$, and $y_3$, using that $(\tan x)' = \sec^2 x$ and $(\sec x)' = \sec x \tan x$:

$$\begin{aligned} y_1' &= 2\sec^2 x \tan x, & y_1'' &= 2((2\sec^2 x\tan^2 x) + \sec^4 x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \\ y_2' &= -4\sec^2 x \tan x, & y_2'' &= -8\sec^2 x \tan^2 x - 4\sec^4 x \\ y_3' &= 2\sec^2 x \tan x, & y_3'' &= 4\sec^2 x\tan^2 x + 2\sec^4 x \end{aligned}$$

The Wronskian of these solutions is then the $3 \times 3$ determinant

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \\ &= \begin{vmatrix} 1+\tan^2 x & 3 - 2\tan^2 x & \sec^2 x \\ 2\sec^2 x\tan x & -4\sec^2 x \tan x & 2\sec^2 x \tan x \\ 4\sec^2 x\tan^2 x + 2\sec^4 x & -8\sec^2 x\tan^2 x - 4\sec^4 x & 4\sec^2 x \tan^2 x + 2\sec^4 x \end{vmatrix} \end{aligned}$$

We expand this determinant along the first row, but first we use the properties of the deteriminant to extract the common factors of the second and third rows:

$$\begin{aligned} W(x) &= (2\sec^2 x\tan x)(4\sec^2 x\tan^2 x + 2\sec^4 x)\begin{vmatrix} 1+\tan^2 x & 3-2\tan^2 x & \sec^2 x \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{vmatrix} \\ &= (8\sec^4 x\tan^3 x + 4\sec^6 x\tan x)\left((1+\tan^2)\begin{vmatrix} -2 & 1 \\ -2 & 1 \end{vmatrix}\right. \\ &\qquad\qquad \left. -(3-2\tan^2 x)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \sec^2 x\begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix}\right) \\ &= (8\sec^4 x\tan^3 x + 4\sec^6 x\tan x)(0-0+0) = 0. \end{aligned}$$

Since the Wronskian is identically 0, the functions are linearly dependent.

**7.** Find the form of a particular solution to the DE $y^{(3)} - 2y'' + 2y' = 6e^x + 3e^x \sin x - x^2$, but do not determine the values of the coefficients.

*Solution*: We first determine the complementary solution $y_c$, the general solution of the associated homogeneous equation. The characteristic equation is $r^3 - 2r^2 + 2r = r(r^2 - 2r + 2) = 0$. Thus $r = 0$ is a root, as are $r = \frac{2\pm\sqrt{4-8}}{2} = 1 \pm i$, so

$$y_c = c_1 + c_2 e^x \cos x + c_3 e^x \sin x.$$

Examining the form of the forcing function $f(x)$, we would expect a particular solution of the form

$$Ae^x + Be^x \cos x + Ce^x \sin x + D + Ex + Fx^2,$$

but the $Be^x \cos x$, $Ce^x \sin x$, and $D$ terms overlap with $y_c$ . Therefore, we multiply these terms and the ones related to them by enough powers of $x$ to prevent the overlap:

$$y_p = Ae^x + Bxe^x \cos x + Cxe^x \sin x + Dx + Ex^2 + Fx^3.$$

**8.** Consider the nonhomogeneous DE $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 8x^{3/2} \sin x$.

(a) Verify that $y_1 = x^{-1/2} \cos x$ and $y_2 = x^{-1/2} \sin x$ are linearly independent solutions to the associated homogeneous DE.

*Solution*: We compute the first and second derivatives of these solutions:

$$y_1' = -\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x \quad y_1'' = \frac{3}{4}x^{-5/2}\cos x + x^{-3/2}\sin x - x^{-1/2}\cos x$$

$$y_2' = -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x \quad y_2'' = \frac{3}{4}x^{-5/2}\sin x - x^{-3/2}\cos x - x^{-1/2}\sin x$$

We plug these into the associated homogeneous DE $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$:

$$\begin{aligned}
&x^2y_1'' + xy_1' + \left(x^2 - \frac{1}{4}\right) y_1 \\
&= x^2 \left(\frac{3}{4}x^{-5/2}\cos x + x^{-3/2}\sin x - x^{-1/2}\cos x\right) \\
&\qquad + x\left(-\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x\right) + \left(x^2 - \frac{1}{4}\right)\left(x^{-1/2}\cos x\right) \\
&= \left(\frac{3}{4} - \frac{1}{2} - \frac{1}{4}\right) x^{-1/2}\cos x + (1-1)\,x^{1/2}\sin x + (-1+1)\,x^{3/2}\cos x = 0
\end{aligned}$$

$$\begin{aligned}
&x^2y_2'' + xy_2' + \left(x^2 - \frac{1}{4}\right) y_2 \\
&= x^2 \left(\frac{3}{4}x^{-5/2}\sin x - x^{-3/2}\cos x - x^{-1/2}\sin x\right) \\
&\qquad + x\left(-\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x\right) + \left(x^2 - \frac{1}{4}\right)\left(x^{-1/2}\sin x\right) \\
&= \left(\frac{3}{4} - \frac{1}{2} - \frac{1}{4}\right) x^{-1/2}\sin x + (-1+1)\,x^{1/2}\cos x + (-1+1)\,x^{3/2}\sin x = 0
\end{aligned}$$

We also check their linear independence by computing the Wronskian:

$$\begin{aligned}
W(x) &= \begin{vmatrix} x^{-1/2}\cos x & x^{-1/2}\sin x \\ -\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x & -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x \end{vmatrix} \\
&= -\frac{1}{2}x^{-2}\sin x\cos x + x^{-1}\cos^2 x + \frac{1}{2}x^{-2}\sin x\cos x + x^{-1}\sin^2 x = \frac{1}{x}
\end{aligned}$$

Since this function is not identically 0, the functions are linearly independent.

(b) Use variation of parameters to find a general solution to the nonhomogeneous DE. (Hint: Use the identities $\sin x\cos x = \frac{1}{2}\sin 2x$ and $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.)

*Solution*: We use the formula of variation of parameters to determine a particular solution to the nonhomogeneous DE. First, we must normalize the DE, so that $y''$ has a coefficient of 1:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right) y = 8x^{-1/2}\sin x.$$

Then $f(x) = 8x^{-1/2}\sin x$, so the formula for variation of parameters is

$$\begin{aligned}
y &= -y_1 \int \frac{y_2 f(x)}{W(x)}\,dx + y_2 \int \frac{y_1 f(x)}{W(x)}\,dx \\
&= -x^{-1/2}\cos x \int \frac{(x^{-1/2}\sin x)(8x^{-1/2}\sin x)}{1/x}\,dx \\
&\qquad + x^{-1/2}\sin x \int \frac{(x^{-1/2}\cos x)(8x^{-1/2}\sin x)}{1/x}\,dx
\end{aligned}$$

Simplifying and applying the identities $\sin x \cos x = \frac{1}{2}\sin 2x$, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, and $\cos 2x = 2\cos^2 x - 1$,

$$\begin{aligned} y &= -x^{-1/2}\cos x \int 8\sin^2 x\, dx + x^{-1/2}\sin x \int 8 \sin x \cos x\, dx \\ &= -x^{-1/2}\cos x \int 4(1-\cos 2x)\, dx + x^{-1/2}\sin x \int 4\sin 2x\, dx \\ &= -x^{-1/2}\cos x(4x - 2\sin 2x) + x^{-1/2}\sin x(-2\cos 2x) \\ &= -x^{-1/2}\cos x(4x - 4\sin x\cos x) + x^{-1/2}\sin x(-2(2\cos^2 x - 1)) \\ &= -4x^{1/2}\cos x + 2x^{-1/2}\sin x. \end{aligned}$$

The second term is already part of the complementary solution, so we can reduce this particular solution to $y = -4x^{1/2}\cos x$.

**9.** A spring is stretched 6 inches by a mass $m$ that weighs 8 lb. The mass is attached to a dashpot that has a damping force of 2 lb-s/ft, and an external force of $4\cos 2t$ lb acts on it.

(a) Describe the steady state response of the system (that is, the particular solution to the nonhomogeneous equation).

*Solution*: We compute the parameters associated to the system: the mass is $m = 8/32 = 1/4$ slug, assuming $g = 32$ ft/s$^2$, and the spring constant is $k = 8/(0.5) = 16$ lb/ft. The damping constant is $c = 2$, as given above. On the forcing side, the amplitude is $F_0 = 4$, and the frequency is $\omega = 2$.

Since the system is damped, the pure-sinusoidal forcing function will not overlap with the transient solution, so we expect the steady-state function will be of the form

$$x(t) = A\cos 2t + B\sin 2t.$$

Plugging this and its derivatives into the nonhomogeneous DE $mx'' + cx' + kx = 4\cos 2t$, we have

$$(A(k-4m) + 2Bc)\cos 2t + ((k-4m)B - 2Ac)\sin 2t = 4\cos 2t.$$

Then $A(k-4m) + 2Bc = 4$ and $(k-4m)B - 2Ac = 0$, which we solve to obtain

$$A = \frac{4(k-4m)}{(k-4m)^2 + 4c^2}, \quad B = \frac{8c}{(k-4m)^2 + 4c^2}.$$

Plugging in $k = 16$, $m = \frac{1}{4}$, and $c = 2$, $k - 4m = 15$ and $2c = 4$, $(k-4m)^2 + 4c^2 = 15^2 + 16 = 241$, so $A = 60/241$ and $B = 16/241$. Then the steady state solution is

$$x(t) = \frac{60}{241}\cos 2t + \frac{16}{241}\sin 2t.$$

We may also unify this into a single oscillation $C\cos(2t-\alpha)$, with

$$C = \sqrt{A^2+B^2} = \frac{4}{\sqrt{(k-4m)^2+4c^2}}, \quad \tan\alpha = \frac{B}{A} = \frac{2c}{k-4m}.$$

Then

$$C = \frac{4}{\sqrt{255+16}} = \frac{4}{\sqrt{241}}, \qquad \alpha = \tan^{-1}\frac{4}{15}, \qquad x(t) = \frac{4}{\sqrt{241}}\cos(2t-\alpha).$$

(b) Find the value of the mass $m$ that maximizes the amplitude of this response, with all other parameters remaining constant. What is this maximum amplitude? What does this mass weigh in pounds?

*Solution*: We see that, as a function of $m$, the amplitude $C$ is maximized when the denominator, or its square

$$(k-4m)^2 + 4c^2,$$

is minimized. Since this is a sum of two squares, only one of which contains $m$, this is minimized when that term is 0. That occurs when $k-4m=0$, so $m=k/4=16/4=4$ slug. This mass weighs 128 lb, and the corresponding amplitude is $C=4/c\omega=4/4=1$ ft.