Homework #1 Solutions

Problems

- Section 1.1: 1, 4, 6, 34, 40
- Section 1.2: 1, 4, 8, 30, 42
- Section 1.4: 1, 2, 3, 4, 8, 22, 24, 46

1.1.1. Verify that $y = x^3 + 7$ is a solution to $y' = 3x^2$.

Solution: From this y, we compute $y' = 3x^2$, which matches the DE. ■

1.1.4. Verify that $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are solutions to $y'' = 9y$.

Solution: We check y_1 and y_2 individually:

- Computing derivatives, $y_1' = 3e^{3x}$ and $y_1'' = 3(3e^{3x}) = 9e^{3x}$, which is the same as $9y = 9e^{3x}$.
- Computing derivatives, $y_2' = -3e^{-3x}$ and $y_2'' = -3(-3e^{-3x}) = 9e^{-3x}$, which is the same as $9y = 9e^{-3x}$. ■

1.1.6. Verify that $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$ are solutions to $y'' + 4y' + 4y = 0$.

Solution: We check y_1 and y_2 individually:

- Computing derivatives, $y_1' = -2e^{-2x}$ and $y_1'' = -2(-2e^{-2x}) = 4e^{-2x}$. Then $y'' + 4y' + 4y = 4e^{-2x} + 4(-2e^{-2x}) + 4(e^{-2x}) = 0$.
- Computing derivatives via the product rule,

$$y_1' = -2xe^{-2x} + e^{-2x} = (1 - 2x)e^{-2x},$$
$$y_1'' = -2(1 - 2x)e^{-2x} - 2e^{-2x} = (4x - 4)e^{-2x}.$$

Then $y'' + 4y' + 4y = (4x - 4)e^{-2x} + 4(1 - 2x)e^{-2x} + 4xe^{-2x} = 0$. ■

1.1.34. Write a differential equation that is a mathematical model of the following situation: The acceleration of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.

Solution: Let v be the velocity of the car, in km/h, and let t be the time, in hours. Then the acceleration is $\frac{dv}{dt}$, which should be proportional to $250 - v$ in the units we have chosen. Thus, the DE is

$$\frac{dv}{dt} = k(250 - v),$$

where k is a constant of proportionality. ∎

1.1.40. Determine by inspection at least one solution to the DE $(y')^2 + y^2 = 1$, then check your solution.

Solution: Answers may vary, but a general solution is $y = \sin(x + C)$, which forms the general solution with the addition of the singular solutions $y = 1$ and $y = -1$. The latter are easy to verify as solutions. For $y = \sin(x + C)$, $y' = \cos(x + C)$, and

$$(y')^2 + y^2 = \cos^2(x + C) + \sin^2(x + C) = 1$$

by the Pythagorean identity. ∎

1.2.1. Find a function $y = f(x)$ satisfying the DE $\frac{dy}{dx} = 2x + 1$ and the IC $y(0) = 3$.

Solution: Integrating, we find the general solution $y = \int 2x + 1\,dx = x^2 + x + C$. Applying the initial condition, $3 = y(0) = 0^2 + 0 + C = C$, so $C = 3$. The particular solution is then $y = x^2 + x + 3$. ∎

1.2.4. Find a function $y = f(x)$ satisfying the DE $\frac{dy}{dx} = \frac{1}{x^2}$ and the IC $y(1) = 5$.

Solution: Integrating, we find the general solution $y = \int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$. Applying the initial condition, $5 = y(1) = -\frac{1}{1} + C = C - 1$. Then $C = 5 + 1 = 6$, so $y = 6 - \frac{1}{x}$. ∎

1.2.8. Find a function $y = f(x)$ satisfying the DE $\dfrac{dy}{dx} = \cos 2x$ and the IC $y(0) = 1$.

Solution: Integrating, we find the general solution $y = \int \cos 2x \, dx = \frac{1}{2}\sin(2x) + C$. Applying the initial condition, $1 = y(0) = \frac{1}{2}\sin(2(0)) + C = 0 + C = C$. Then $C = 1$, so $y = \frac{1}{2}\sin(2x) + 1$. ∎

1.2.30. A car traveling at 60 mi/h (88 ft/s) skids 176 ft after its brakes are suddenly applied. Under the assumption that the braking system provides constant deceleration, what is that deceleration? For how long does the skid continue?

Solution: In general, we expect that the position and velocity of the car are given by $x(t) = \frac{1}{2}at^2 + v_0 t + x_0$ and $v(t) = at + v_0$, where $v(0) = v_0$ and $x(0) = x_0$ are the initial velocity and position of the car. In this case, $v_0 = 88$, using units of feet for x and seconds for t. We also set $x_0 = 0$.

Let T denote the stopping time for the car, and let $L = 176$ denote the length of the skid, in feet. Then $v(T) = 0$, and $x(T) = L$. We now must solve

$$v(T) = aT + v_0 = 0, \qquad x(T) = \frac{1}{2}aT^2 + v_0 T = L$$

for a and T simultaneously. From the first equation, $aT = -v_0$, so $T = -\frac{v_0}{a}$. Substituting this into the second equation, $\frac{1}{2}\frac{v_0^2}{a} - \frac{v_0^2}{a} = L$, so $v_0^2 = -2aL$, and $a = -\frac{v_0^2}{2L}$. Thus, $T = -v_0(-\frac{2L}{v_0^2}) = \frac{2L}{v_0}$. Plugging in the values $v_0 = 88$ and $L = 176$,

$$a = -\frac{(88)^2}{2(176)} = -\frac{88}{4} = -22\,\frac{\text{ft}}{\text{s}^2}, \qquad T = \frac{2(176)}{88} = 4 \text{ s.}$$ ∎

1.2.42. A spacecraft is in free fall towards the surface of the moon at a speed of 1000 mph (mi/h). Its retrorockets, when fired, provide a constant deceleration of 20,000 mi/h^2. At what height above the lunar surface should the astronauts fire the retrorockets to insure a safe touchdown? (As in Example 1.2.2, ignore the moon's gravitational field.)

Solution: Let y denote the height of the spacecraft above the surface of the moon, and suppose that the spacecraft fires its retrorockets at $t = 0$. Then, in mi/h, $v_0 = -1000$, since the spacecraft is descending, and, in mi/h^2, $a = 20{,}000$, since the retrorockets provide an upward force. The velocity $v(t)$ and height $y(t)$ of the spacecraft are then

$$v(t) = at + v_0, \qquad y(t) = \frac{1}{2}at^2 + v_0 t + y_0,$$

where y_0 is then the height of the craft when it starts to fire the rockets.

Let $t = L$ be the time at which the craft lands. For a soft landing, we expect to have $v(L) = 0$ and $x(L) = 0$, so

$$v(L) = aL + v_0 = 0, \qquad y(L) = \frac{1}{2}aL^2 + v_0 L + y_0 = 0.$$

Solving the first equation, $L = -v_0/a = 1000/20{,}000 = 0.05$ h. We plug this into the second equation to get

$$0 = \frac{1}{2}(20{,}000)(0.05)^2 - 1000(0.05) + y_0 = 25 - 50 + y_0 = y_0 - 25,$$

so $y_0 = 25$ mi. Thus, the craft should start firing its rockets 25 miles above the surface to land softly. ∎

1.4.1. Find the general solution (implicit if necessary, explicit if possible) to the DE $\frac{dy}{dx} + 2xy = 0$.

Solution: Writing the equation in normal form, we obtain $y' = -2xy$, which is separable. Assuming $y \neq 0$, we separate variables to form $\frac{1}{y}y' = -2x$. Integrating, we obtain

$$\int \frac{1}{y}\,dy = \int -2x\,dx$$
$$\ln|y| = -x^2 + C,$$

where C is any real number. Then $y = \pm e^{-x^2+C} = \pm e^C e^{-x^2}$. We observe that $\pm e^C$ is just another constant, and that its range of values is all real numbers except 0. We therefore redefine C to be this value, so that $y = Ce^{-x^2}$, $C \neq 0$.

This family is one set of solutions to the DE, but because of the assumptions made when separating variables, we may have missed $y = 0$ as a solution. Fortunately, $y = 0 = 0e^{-x^2}$ exactly corresponds to the one value of C that we excluded, so our general solution is $y = Ce^{-x^2}$, where C is any real.

We also check that this general solution is indeed valid: $y' = -2xCe^{-x^2} = -2xy$, for all values of C. ∎

1.4.2. Find the general solution (implicit if necessary, explicit if possible) to the DE $\frac{dy}{dx} + 2xy^2 = 0$.

Solution: Writing the equation in normal form, we obtain $y' = -2xy^2$, which is separable. Assuming $y \neq 0$, we separate variables to form $\frac{1}{y^2}y' = -2x$. Integrating, we obtain

$$\int \frac{1}{y^2}\,dy = \int -2x\,dx$$
$$-\frac{1}{y} = -x^2 + C,$$

where C is any real number. Then $\frac{1}{y} = x^2 + C$ (reversing the sign on C), so $y = \frac{1}{x^2+C}$ is a general solution. We check that it is a valid solution:

$$y' = (-1)(2x)(x^2 + C)^{-2} = -2x\left(\frac{1}{x^2 + C}\right)^2 = -2xy^2.$$

Furthermore, we must go back and check the potential solution $y = 0$, which we excluded algebraically when separating. This, too, is a valid solution: $y' = 0 = -2x(0)^2 = -2xy^2$. Additionally, there is no value of C so that $\frac{1}{x^2+C} = 0$ as functions of x, so this solution remains separate from the other family of solutions.

Thus, the complete general solution to the DE is $y = \frac{1}{x^2+C}$ or $y = 0$. (Actually, in a sense, $y = 0$ is the limit of the family as $C \to \infty$, but we do not make that precise in this course.) ■

1.4.3. Find the general solution (implicit if necessary, explicit if possible) to the DE $\frac{dy}{dx} = y \sin x$.

Solution: The equation is already in normal form, and is separable, so we separate the variables to form $\frac{1}{y}y' = \sin x$. We observe this will exclude the potential solution $y = 0$. Intergrating,

$$\int \frac{1}{y}\,dy = \int \sin x\,dx \qquad \Rightarrow \qquad \ln|y| = -\cos x + C$$

Then $y = \pm e^C e^{-\cos x} = Ce^{-\cos x}$, where C (redefined) is nonzero. We check that this is a solution: $y' = C(\sin x)e^{-\cos x} = y \sin x$.

We also note that if $y = 0$, $y' = 0 = (0)(\sin x) = y \sin x$, so this is an additional solution to the DE that corresponds to the general solution above when $C = 0$. Thus, the general solution to the DE is $y = Ce^{-\cos x}$, C any real. ■

1.4.4. Find the general solution (implicit if necessary, explicit if possible) to the DE $(1 + x)\frac{dy}{dx} = 4y$.

Solution: We observe that this DE is separable, so we form $\frac{1}{y}y' = \frac{4}{1+x}$ and integrate:

$$\int \frac{1}{y}\,dy = \int \frac{4}{1+x}\,dx \qquad \Rightarrow \qquad \ln|y| = 4\ln|1 + x| + C = \ln(1 + x)^4 + C$$

Then $y = C(1 + x)^4$; including the omitted solution $y = 0$ as corresponding to $C = 0$, C can be any real number. We check that these are solutions to the original DE: $y' = 4C(1 + x)^3$, so

$$(1 + x)y' = 4C(1 + x)^4 = 4y.$$

■

1.4.8. Find the general solution (implicit if necessary, explicit if possible) to the DE $\frac{dy}{dx} = 2x \sec y$.

Solution: Separating, we obtain $\frac{1}{\sec y} y' = 2x$, or $(\cos y) y' = 2x$. Then integrating,

$$\int \cos y \, dy = \int 2x \, dx \qquad \Rightarrow \qquad \sin y = x^2 + C.$$

Then $y = \sin^{-1}(x^2 + C)$. We actually omitted no solutions during the separation process, as sec is never 0. The domain on these solutions is restricted to $-1 \leq x^2 + C \leq 1$; for $C > 1$, the function is not defined at all.

For good measure, we check that this general solution is valid: by the chain rule,

$$y' = (2x) \frac{1}{\sqrt{1 - (x^2 + C)^2}} = \frac{2x}{\cos y} = 2x \sec y.$$

■

1.4.22. Find an explicit particular solution of the initial value problem $\frac{dy}{dx} = 4x^3 y - y$, $y(1) = -3$.

Solution: We separate variables to obtain $\frac{1}{y} y' = 4x^3 - 1$. Integrating,

$$\int \frac{1}{y} \, dy = \int 4x^3 - 1 \, dx \qquad \Rightarrow \qquad \ln |y| = x^4 - x + C.$$

Then $y = Ce^{x^4 - x}$ for any real C, with the solution $y = 0$ filling the $C = 0$ hole missed during integration.

We apply the initial condition: $-3 = Ce^{1-1} = C(1) = C$, so $C = -3$, and the solution is $y = -3e^{x^4 - x}$. We also check that this is valid for the original DE:

$$y' = -3(4x^3 - 1)e^{x^4 - x} = (4x^3 - 1)y = 4x^3 y - y.$$

■

1.4.24. Find an explicit particular solution of the initial value problem $(\tan x)\frac{dy}{dx} = y$, $y(\frac{1}{2}\pi) = \frac{1}{2}\pi$.

Solution: We separate variables to obtain $\frac{1}{y} y' = \cot x = \frac{\cos x}{\sin x}$. Integrating,

$$\int \frac{1}{y} \, dy = \int \frac{\cos x}{\sin x} \, dx \qquad \Rightarrow \qquad \ln |y| = \ln |\sin x| + C.$$

Then $y = C|\sin x|$ for any real C, with the solution $y = 0$ filling the $C = 0$ hole missed during integration. In fact, the solutions are unique only away from $x = n\pi$, where

$\sin x = 0$, so we may write each solution as $C \sin x$, since $\sin x$ has a single sign on each interval $(n\pi, (n+1)\pi)$.

We apply the initial condition: $\frac{1}{2}\pi = C \sin(\frac{1}{2}\pi) = C(1) = C$, so $C = \frac{1}{2}\pi$, and the solution is $y = \frac{1}{2}\pi \sin x$. We also check that this is valid for the original DE: $y' = \frac{1}{2}\pi \cos x$, so

$$(\tan x)y' = \frac{\sin x}{\cos x}\left(\frac{1}{2}\pi \cos x\right) = \frac{1}{2}\pi \sin x = y.$$

We note that the DE initially does not appear to be meaningful at $x = \frac{\pi}{2}$, since $\tan x$ is not defined there, but in some sense this is a "removable discontinuity" that disappears when the equation is put into normal form as $y' = y \cot x$. ∎

1.4.46. The barometric pressure p (in inches of mercury) at an altitude x miles above sea level satisfies the initial value problem $\frac{dp}{dx} = -0.2p$, $p(0) = 29.92$.

(a) Calculate the barometric pressure at 10,000 ft and again at 30,000 ft.

(b) Without prior conditioning, few people can survive when the pressure drops to less than 15 inches of mercury. How high is that?

Solution (a): The DE is one of the form $p' = kp$, with $k = -0.2/\text{mi}$, and so has the solution $p(x) = p(0)e^{-0.2x} = 29.92e^{-0.2x}$, with x in miles. Then, since there are 5280 feet in a mile,

$$\begin{aligned} p(10{,}000 \text{ ft}) &= 29.92e^{-0.2(10{,}000/5280)} = 20.49 \text{ in Hg} \\ p(30{,}000 \text{ ft}) &= 29.92e^{-0.2(30{,}000/5280)} = 9.60 \text{ in Hg}. \end{aligned}$$

∎

Solution (b): We solve $29.92e^{-0.2x} = 15$:

$$x = -\frac{1}{0.2} \ln \frac{15}{29.92} = -5 \ln \frac{15}{29.92} \approx 3.45 \text{ mi} \approx 18{,}200 \text{ ft}.$$

∎