Homework #5 Solutions

Problems

- Section 2.4: 6, 8
- Section 3.1: 4, 6, 10, 14, 18

2.4.6. Given the IVP $y' = -2xy$, $y(0) = 2$, apply Euler's method to approximate a solution on the interval $[0, 1/2]$, first with step size $h = 0.25$, and then with step size $h = 0.1$. Compare the three-decimal-place values of the two approximations at $x = 1/2$ with the value $y(1/2)$ of the actual solution, $y(x) = 2e^{-x^2}$.

Solution: Computing $x_n = x_{n-1} + h$ and $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$ iteratively with $f(x, y) = -2xy$ and step sizes $h = 0.25$ and $h = 0.1$,

n	x_n	y_n
0	0	2
1	0.25	2
2	0.5	1.75

n	x_n	y_n
0	0	2
1	0.1	2
2	0.2	1.96
3	0.3	1.882
4	0.4	1.769
5	0.5	1.627

Therefore, the approximate values are 1.750 and 1.627, respectively. The actual value is $e^{-1/4} \approx 1.558$, so the absolute and relative errors are

$$AE_{0.25} = |1.750 - 1.558| = 0.192 \qquad RE_{0.25} = \frac{AE_0.25}{1.558} = 0.124 = 12.4\%$$

$$AE_{0.25} = |1.627 - 1.558| = 0.069 \qquad RE_{0.25} = \frac{AE_0.25}{1.558} = 0.045 = 4.5\% \qquad \blacksquare$$

2.4.8. Given the IVP $y' = e^{-y}$, $y(0) = 0$, apply Euler's method to approximate a solution on the interval $[0, 1/2]$, first with step size $h = 0.25$, and then with step size $h = 0.1$. Compare the three-decimal-place values of the two approximations at $x = 1/2$ with the value $y(1/2)$ of the actual solution, $y(x) = \ln(x + 1)$.

Solution: Computing $x_n = x_{n-1} + h$ and $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$ iteratively with $f(x, y) = e^{-y}$ and step sizes $h = 0.25$ and $h = 0.1$,

n	x_n	y_n
0	0	0
1	0.25	0.25
2	0.5	0.445

n	x_n	y_n
0	0	0
1	0.1	0.1
2	0.2	0.190
3	0.3	0.273
4	0.4	0.349
5	0.5	0.420

$\blacksquare$

Therefore, the approximate values are 0.445 and 0.420, respectively. The actual value is $\ln(1.5) \approx 0.405$, so the absolute and relative errors are

$$AE_{0.25} = |0.445 - 0.405| = 0.040 \qquad RE_{0.25} = \frac{AE_0.25}{0.405} = 0.097 = 9.7\%$$
$$AE_{0.25} = |0.420 - 0.405| = 0.015 \qquad RE_{0.25} = \frac{AE_0.25}{0.405} = 0.037 = 3.7\%$$

3.1.4. Verify that $y_1 = \cos 5x$ and $y_2 = \sin 5x$ are solutions to $y'' + 25y = 0$. Find a particular solution of the form $y = c_1y_1 + c_2y_2$ so that $y(0) = 10$ and $y'(0) = -10$.

Solution: We first compute the derivatives of these solutions:

$$y_1' = -5\sin 5x \qquad y_2' = 5\cos 5x$$
$$y_1'' = -25\cos 5x \qquad y_2'' = -25\sin 5x$$

Then

$$y_1'' + 25y_1 = -25\cos 5x + 25\cos 5x = 0$$
$$y_2'' + 25y_2 = -25\sin 5x + 25\sin 5x = 0$$

so both are solutions. To solve the IVP, we set up the system

$$y(0) = c_1\cos(5(0)) + c_2\sin(5(0)) = 10 \quad y'(0) = -5c_1\sin(5(0)) + 5c_2\cos(5(0)) = -10.$$

Then $c_1 = 10$, and $5c_2 = -10$, so $c_2 = -2$. Therefore, $y = 10\cos 5x - 2\sin 5x$. ■

3.1.6. Verify that $y_1 = e^{2x}$ and $y_2 = e^{-3x}$ are solutions to $y'' + y' - 6y = 0$. Find a particular solution of the form $y = c_1y_1 + c_2y_2$ so that $y(0) = 7$ and $y'(0) = -1$.

Solution: We first compute the derivatives of these solutions:

$$y_1' = 2e^{2x} \qquad y_2' = 4e^{2x}$$
$$y_1'' = -3e^{-3x} \qquad y_2'' = 9e^{-3x}$$

Then

$$y_1'' + y_1' - 6y_1 = 4e^{2x} + 2e^{2x} - 6e^{2x} = 0$$
$$y_2'' + y_2' - 6y_2 = 9e^{-3x} - 3e^{-3x} + 6e^{-3x} = 0$$

so both are solutions. To solve the IVP, we set up the system

$$y(0) = c_1e^{2(0)} + c_2e^{-3(0)} = 7 \qquad y'(0) = 2c_1e^{2(0)} - 3c_2e^{-3(0)} = -1.$$

Then $c_1 + c_2 = 7$, and $2c_1 - 3c_2 = -1$. Solving this linear system, $c_1 = 4$ and $c_2 = 3$, so $y = 4e^{2x} + 3e^{-3x}$. ■

3.1.10. Verify that $y_1 = e^{5x}$ and $y_2 = xe^{5x}$ are solutions to $y'' - 10y' + 25y = 0$. Find a particular solution of the form $y = c_1 y_1 + c_2 y_2$ so that $y(0) = 3$ and $y'(0) = 13$.

Solution: We first compute the derivatives of these solutions:

$$y_1' = 5e^{5x} \qquad y_2' = 25e^{5x}$$
$$y_1'' = e^{5x} + 5xe^{5x} = (1+5x)e^{5x} \qquad y_2'' = 5e^{5x} + 5(1+5x)e^{5x} = (10+25x)e^{5x}$$

Then

$$y_1'' - 10y_1' + 25y_1 = (25 - 50 + 25)e^{5x} = 0$$
$$y_2'' - 10y_2' + 25y_2 = (10 + 25x - 10 - 50x + 25x)e^{5x} = 0$$

so both are solutions. To solve the IVP, we set up the system

$$y(0) = c_1 e^{5(0)} + c_2(0)e^{5(0)} = 3 \qquad y'(0) = 5c_1 e^{5(0)} + c_2(1 + 5(0))e^{5(0)} = 13.$$

Then $c_1 = 3$, and $5c_1 + c_2 = 13$, so $c_2 = 13 - 15 = -2$. Thus, the solution is $y = 3e^{5x} - 2xe^{5x}$. ∎

3.1.14. Verify that $y_1 = x^2$ and $y_2 = x^{-3}$ are solutions to $x^2y'' + 2xy' - 6y = 0$. Find a particular solution of the form $y = c_1 y_1 + c_2 y_2$ so that $y(2) = 10$ and $y'(2) = 15$.

Solution: We first compute the derivatives of these solutions:

$$y_1' = 2x \qquad y_2' = 2$$
$$y_1'' = -3x^{-4} \qquad y_2'' = 12x^{-5}$$

Then

$$x^2y_1'' + 2xy_1' - 6y_1 = 2x^2 + 4x^2 - 6x^2 = 0$$
$$x^2y_2'' + 2xy_2' - 6y_2 = 12x^{-2} - 6x^{-2} - 6x^{-2} = 0$$

so both are solutions. To solve the IVP, we set up the system

$$y(2) = c_1 2^2 + c_2 2^{-3} = 10 \qquad y'(2) = 2c_1(2) - 3c_2 2^{-4} = 15.$$

Then $4c_1 + c_2/8 = 10$, and $4c_1 - 3c_2/16 = 15$. Subtracting the second equation from the first, $5c_2/16 = -5$, so $c_2 = -16$, and $c_1 = 3$. Thus, $y = 3x^2 - 16x^{-3}$. ∎

3.1.18. Show that $y = x^3$ is a solution of $yy'' = 6x^4$, but that if $c^2 \neq 1$, then $y = cx^3$ is not a solution.

Solution: Letting $y = cx^3$, $y' = 3cx^2$, and $y'' = 6cx$. Plugging these into the DE, $yy'' = (cx^3)(6cx) = 6c^2x^4$. Thus, this is equal to $6x^4$ if and only if $c^2 = 1$. Therefore, $y = x^3$ and $y = -x^3$ are the only solutions to the DE of this form. ∎