Homework #11 Solutions

Problems

- Section 5.2: 10, 12, 24, 28. Omit the graphing on problems 10 and 12.
- Section 5.4: 2, 6, 12. Omit the graphing on problems 2 and 6.
- Additional Problem #1.

5.2.10. Apply the eigenvalue method to find a general solution to the system $x_1' = -3x_1 - 2x_2$, $x_2' = 9x_1 + 3x_2$.

Solution: We write the system as $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad A = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix}.$$

We compute the eigenvalues of A from $\det(A - \lambda I)$:

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -2 \\ 9 & 3-\lambda \end{vmatrix} = \lambda^2 - 9 + 18 = \lambda^2 + 9.$$

Thus, the eigenvalues are the pure imaginary pair $\lambda = \pm 3i$. Undaunted, we compute eigenvectors for one of the eigenvalues in this pair, $\lambda = 3i$. Row reducing $A - 3iI$ gives

$$\begin{bmatrix} -3-3i & -2 \\ 9 & 3-3i \end{bmatrix} \sim \begin{bmatrix} -3-3i & -2 \\ 3 & 1-i \end{bmatrix} \sim \begin{bmatrix} 3 & 1-i \\ 0 & 0 \end{bmatrix},$$

where we use that $(1+i)\begin{bmatrix} 3 & 1-i \end{bmatrix} = \begin{bmatrix} 3+3i & 2 \end{bmatrix}$ is the opposite of the top row. Hence, one choice of complex eigenvector is $\mathbf{v} = \begin{bmatrix} 1-i \\ -3 \end{bmatrix}$, and one complex-valued solution is

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v} = e^{3ti}\mathbf{v} = (\cos 3t + i \sin 3t)\begin{bmatrix} 1-i \\ -3 \end{bmatrix} = \begin{bmatrix} \cos 3t + i \sin 3t - i \cos 3t + \sin 3t \\ -3\cos 3t - 3i \sin 3t \end{bmatrix}$$

We get another linearly independent solution from $\lambda = -3i$, but we may take its eigenvector to be the complex conjugate of $\mathbf{v}$, and therefore this solution is also the complex conjugate of $\mathbf{x}(t)$ above. Hence, we may isolate the real and complex parts of $\mathbf{x}(t)$ as real-valued, linearly independent solutions to the DE:

$$\mathbf{x}_1(t) = \begin{bmatrix} \cos 3t + \sin 3t \\ -3\cos 3t \end{bmatrix}, \qquad \mathbf{x}_2(t) = \begin{bmatrix} \sin 3t - \cos 3t \\ -3 \sin 3t \end{bmatrix}.$$

Therefore, the general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = \begin{bmatrix} c_1(\cos 3t + \sin 3t) + c_2(\sin 3t - \cos 3t) \\ -3c_1 \cos 3t - 3c_2 \sin 3t \end{bmatrix}.$$

Note that the complex eigenvector $\mathbf{v}$ is determined up to a complex scalar multiple, the choice of which may drastically change the linear combinations of sin and cos terms in the final solution. Nevertheless, all such forms should be equivalent, and are related by different, invertible linear combinations of the coefficients c_1 and c_2. ∎

5.2.12. Apply the eigenvalue method to find a general solution to the system $x_1' = x_1 - 5x_2$, $x_2' = x_1 + 3x_2$.

Solution: We write the system as $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & -5 \\ 1 & 3 \end{bmatrix}.$$

We compute the eigenvalues of A from $\det(A - \lambda I)$:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 8.$$

Thus, the eigenvalues are the complex pair $\lambda = 2 \pm 2i$. We again compute an eigenvector for one of these eigenvalues, say $\lambda = 2 + 2i$:

$$A - \lambda I = \begin{bmatrix} -1-2i & -5 \\ 1 & 1-2i \end{bmatrix} \sim \begin{bmatrix} 1 & 1-2i \\ 0 & 0 \end{bmatrix}$$

Therefore, one choice of eigenvector is $\mathbf{v} = \begin{bmatrix} 1-2i \\ -1 \end{bmatrix}$. Since $e^{\lambda t} = e^{2t}(\cos 2t + i \sin 2t)$, the complex solution is

$$\mathbf{x}(t) = e^{2t}(\cos 2t + i \sin 2t) \begin{bmatrix} 1-2i \\ -1 \end{bmatrix} = e^{2t} \begin{bmatrix} \cos 2t + i \sin 2t - 2i \cos 2t + 2 \sin 2t \\ -\cos 2t - i \sin 2t \end{bmatrix}.$$

Taking linear combinations of the real and imaginary parts gives the general solution as a linear combination of real-valued functions:

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} \cos 2t + 2 \sin 2t \\ -\cos 2t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \sin 2t - 2 \cos 2t \\ -\sin 2t \end{bmatrix}.$$

As above, solutions may vary, but should be equivalent with different choices of linear combinations of the coefficients c_1 and c_2. ∎

5.2.24. Apply the eigenvalue method to find a general solution to the system $x_1' = 2x_1 + x_2 - x_3$, $x_2' = -4x_1 - 3x_2 - x_3$, $x_3' = 4x_1 + 4x_2 + 2x_3$.

Solution: We write the system as $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \qquad A = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix}.$$

We compute the eigenvalues of A from $\det(A - \lambda I)$:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 1 & -1 \\ -4 & -3-\lambda & -1 \\ 4 & 4 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)\begin{vmatrix} -3-\lambda & -1 \\ 4 & 2-\lambda \end{vmatrix} - \begin{vmatrix} -4 & -1 \\ 4 & 2-\lambda \end{vmatrix} + (-1)\begin{vmatrix} -4 & -3-\lambda \\ 4 & 4 \end{vmatrix} \\ &= (2-\lambda)(\lambda^2 + \lambda - 2) - (4\lambda - 8 + 4) - (-16 + 12 + 4\lambda) \\ &= -\lambda^3 - \lambda^2 + 2\lambda + 2\lambda^2 + 2\lambda - 4 - 4\lambda + 4 - 4\lambda + 4 \\ &= -\lambda^3 + \lambda^2 - 4\lambda^3 + 4 = -(\lambda - 1)(\lambda^2 + 4). \end{aligned}$$

Therefore, the eigenvalues are $\lambda = 1$ and the complex pair $\lambda = \pm 2i$. We determine eigenvectors for $\lambda = 1$ and $\lambda = 2i$. First,

$$A - 1I = \begin{bmatrix} 1 & 1 & -1 \\ -4 & -4 & -1 \\ 4 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, we may choose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, so one solution is $\mathbf{x}_1(t) = e^t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Taking $\lambda = 2i$,

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 2-2i & 1 & -1 \\ -4 & -3-2i & -1 \\ 4 & 4 & 2-2i \end{bmatrix} \sim \begin{bmatrix} 0 & -1+2i & -1+2i \\ 0 & 1-2i & 1-2i \\ 1 & 1 & \frac{1}{2} - \frac{1}{2}i \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & \frac{1}{2} - \frac{1}{2}i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} - \frac{1}{2}i \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Then, writing $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $a = (\frac{1}{2} + \frac{1}{2}i)c$ and $b = -c$. Taking $c = 2$, $\mathbf{v} = \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix}$, so a complex-valued solution is

$$\mathbf{x}(t) = e^{i2t}\mathbf{v} = \begin{bmatrix} \cos 2t + i \sin 2t + i \cos 2t - \sin 2t \\ -2\cos 2t - 2i \sin 2t \\ 2\cos 2t + 2i \sin 2t \end{bmatrix}.$$

Therefore, taking its real and imaginary parts, we obtain the general solution

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \cos 2t - \sin 2t \\ -2\cos 2t \\ 2\cos 2t \end{bmatrix} + c_3 \begin{bmatrix} \sin 2t + \cos 2t \\ -2\sin 2t \\ 2\sin 2t \end{bmatrix}$$

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5.2.28. The amounts $x_1(t)$ and $x_2(t)$ of salt in the two brine tanks of Figure 5.2.7 satisfy the differential equation

$$\frac{dx_1}{dt} = -k_1 x_1, \qquad \frac{dx_2}{dt} = k_1 x_1 - k_2 x_2,$$

where $k_i = r/V_i$ for $i = 1, 2$. First, solve for $x_1(t)$ and $x_2(t)$ assuming that $r = 10$ gal/min, $V_1 = 25$ gal, $V_2 = 40$ gal, $x_1(0) = 15$ lb, and $x_2(0) = 0$ lb. Then find the maximum amount of salt ever in tank 2. Finally, graph $x_1(t)$ and $x_2(t)$ together.

Solution: With these values for r, V_1, and V_2, $k_1 = \frac{2}{5}$ and $k_2 = \frac{1}{4}$. Then the system is $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} -\frac{2}{5} & 0 \\ \frac{2}{5} & -\frac{1}{4} \end{bmatrix}$. Since the matrix is lower triangular, we immediately obtain its eigenvalues $\lambda_1 = -\frac{2}{5}$ and $\lambda_2 = -\frac{1}{4}$ from the diagonal coefficients. We then find eigenvectors for these eigenvalues.

$$A + \frac{2}{5}I = \begin{bmatrix} 0 & 0 \\ \frac{2}{5} & \frac{3}{20} \end{bmatrix} \sim \begin{bmatrix} 8 & 3 \\ 0 & 0 \end{bmatrix}, \qquad A + \frac{1}{4}I = \begin{bmatrix} -\frac{3}{20} & 0 \\ \frac{2}{5} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

so we take $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence, the general solution is

$$\mathbf{x}(t) = c_1 e^{-2/5t} \begin{bmatrix} 3 \\ -8 \end{bmatrix} + c_2 e^{-1/4t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To solve the initial condition, we find c_1 and c_2 so that $\mathbf{x}(0) = \begin{bmatrix} 15 \\ 0 \end{bmatrix}$. This produces the augmented system

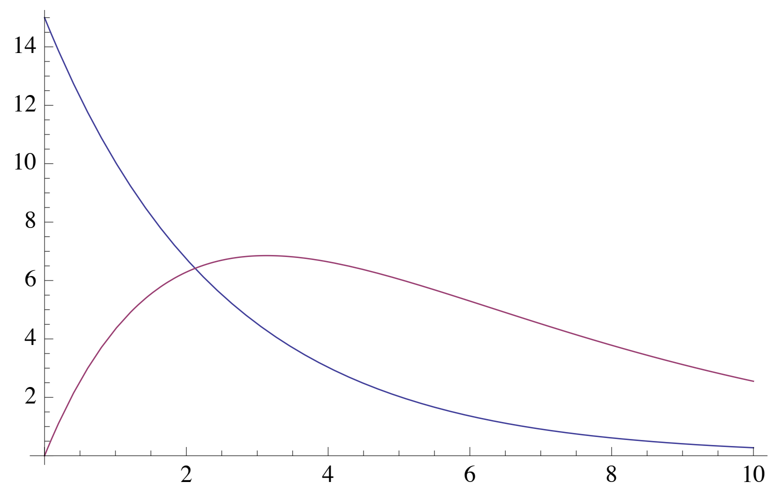
$$\left[\begin{array}{cc|c} 3 & 0 & 15 \\ -8 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 15 \\ 0 & 1 & 40 \end{array}\right],$$

so $c_1 = 5$ and $c_2 = 40$. Then $x_1(t) = 15e^{-3/20t}$ and $x_2(t) = 40(e^{-1/4t} - e^{-2/5t})$.

We find the maximum value of $x_2(t)$. First, we find the t at which that occurs: solving $x_2'(t) = 0$, we have $40(-\frac{1}{4}e^{-1/4t} + \frac{2}{5}e^{-2/5t} = 0$, so $5e^{-1/4t} = 8e^{-2/5t}$, and $t = \frac{20}{3}\ln\frac{8}{5}$. Plugging this back into $x_2(t)$, and simplifying the exponentials with $e^{\ln\frac{8}{5}} = \frac{8}{5}$, we have

$$x_2\left(\frac{20}{3}\ln\frac{8}{5}\right) = 40\left(e^{-\frac{5}{3}\ln\frac{8}{5}} - e^{-\frac{8}{3}\ln\frac{8}{5}}\right) = 40\left(\frac{8}{5}\right)^{-8/3}\left(\frac{8}{5} - 1\right) = \frac{75\sqrt[3]{25}}{32} \approx 6.85 \text{ lb.}$$

Finally, we plot $x_1(t)$ and $x_2(t)$ below for $0 \le t \le 10$:



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> 5.4.2. Find the general solution of the system $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x}$.

Solution: We first compute the eigenvalues of $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0.$$

Then the only eigenvalue is $\lambda = 2$, with multiplicity 2. We find any associated eigenvectors:

$$A - 2I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

so the only eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We therefore look for a generalized eigenvector $\mathbf{v}_2$ so that $(A - 2I)\mathbf{v}_2 = \mathbf{v}_1$, setting up the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Then, writing $\mathbf{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$, $a = b + 1$. Taking $b = 0$, we have $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, although other choices are just as valid, and will differ by a multiple of the genuine eigenvector $\mathbf{v}_1$. In any case, we construct the general solution

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = e^{2t} \begin{bmatrix} c_1 + c_2 t \\ c_1 + c_2(t+1) \end{bmatrix}.$$

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5.4.6. Find the general solution of the system $\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix} \mathbf{x}$.

Solution: We first compute the eigenvalues of $A = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix}$:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0.$$

Then the only eigenvalue is $\lambda = 5$, with multiplicity 2. We find any associated eigenvectors:

$$A - 5I = \begin{vmatrix} -4 & -4 \\ 4 & 4 \end{vmatrix} \sim \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$$

Then there is only one eigenvector, and we may take it to be $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Since we have only one eigenvector, we must find a generalized eigenvector $\mathbf{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$ so that $(A - 5I)\mathbf{v}_2 = \mathbf{v}_1$. Setting this up as an augmented system,

$$\left[\begin{array}{cc|c} -4 & -4 & 1 \\ 4 & 4 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \end{array}\right]$$

Then $a = -b - \frac{1}{4}$, so we may take $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}$ (again with the same potential $\mathbf{v}_1$ indeterminacy). Hence, we can write our general solution as

$$\mathbf{x}(t) = c_1 e^{5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{5t} \left(t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix} \right) = e^{5t} \begin{bmatrix} c_1 + c_2(t - \frac{1}{4}) \\ -c_1 - c_2 t \end{bmatrix}$$ ∎

5.4.12. Find the general solution of the system $\mathbf{x}' = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \mathbf{x}$.

Solution: As above, we first compute the eigenvalues of $A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & -1-\lambda & 1 \\ 1 & -1 & -1-\lambda \end{vmatrix} \\ &= (-1-\lambda) \begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} + (1) \begin{vmatrix} 0 & -1-\lambda \\ 1 & -1 \end{vmatrix} \\ &= -(1+\lambda)((1+\lambda)^2 + 1) - (1+\lambda) = -(1+\lambda)^3 \end{aligned}$$

Then the only eigenvalue is $\lambda = -1$, with multiplicity 3. We find any associated eigenvectors:

$$A + I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the only choice (up to scale) for an eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Hence, we must find two more generalized eigenvectors forming a chain $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of length 3. We find $\mathbf{v}_2$ so that $(A + I)\mathbf{v}_2 = \mathbf{v}_1$:

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then $\mathbf{v}_2 = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so we take $a = 0$ for $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Next, we find $\mathbf{v}_3$ so $(A + I)\mathbf{v}_3 = \mathbf{v}_2$:

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then $\mathbf{v}_3 = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so we take $a = 0$ for $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Consequently, the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \left(t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) + c_3 e^{-t} \left(\frac{t^2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \\ &= e^{-t} \begin{bmatrix} c_1 + c_2 t + \frac{1}{2} c_3 t^2 + c_3 \\ c_1 + c_2 t + \frac{1}{2} c_3 t^2 \\ c_2 + c_3 t \end{bmatrix} \end{aligned}$$

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A.P. #1. In problem 4.1.24, we derived the system

$$m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2, \qquad m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2$$

to model the displacements $x_1(t)$ and $x_2(t)$ of the two masses m_1 and m_2 in the mass-spring system depicted in Figure 4.1.11. Introducing variables $y_1 = x_1'$ and $y_2 = x_2'$ and normalizing, we write this as the first-order system

$$\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}$$

Assume that $m_1 = m_2 = 1$, $k_1 = k_3 = 2$, and $k_2 = 1$.

(a) Use the eigenvalue and eigenvector techniques from section 5.2 to find the general solution of this linear system.

(b) Find the solution matching the initial conditions $x_1(0) = 3$, $x_2(0) = 1$, $x_1'(0) = 0$, and $x_2'(0) = 0$.

(c) Write the $x_1(t)$ and $x_2(t)$ components of your solution in the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \cos(\omega_1 t - \alpha_1)\mathbf{v}_1 + \cos(\omega_2 t - \alpha_2)\mathbf{v}_2$$

for frequencies ω_i, phases α_i, and amplitude vectors $\mathbf{v}_i$. Interpret each of these two terms with respect to the motion of the masses in the system.

Solution (a): With $m_1 = m_2 = 1$, $k_1 = k_3 = 2$, and $k_2 = 1$, the matrix A above is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 \end{bmatrix}$$

We find its eigenvalues, computing $\det(A - \lambda I)$ by row expansion:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -3 & 1 & -\lambda & 0 \\ 1 & -3 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ -3 & 0 & -\lambda \end{vmatrix} + \begin{vmatrix} 0 & -\lambda & 1 \\ -3 & 1 & 0 \\ 1 & -3 & -\lambda \end{vmatrix}$$

$$= (-\lambda)(-\lambda) \begin{vmatrix} -\lambda & 1 \\ -3 & -\lambda \end{vmatrix} + \lambda \begin{vmatrix} -3 & 0 \\ 1 & -\lambda \end{vmatrix} + \lambda \begin{vmatrix} -3 & 1 \\ 1 & -3 \end{vmatrix}$$

$$= \lambda^2(\lambda^2 + 3) + 3\lambda + 8 = \lambda^4 + 6\lambda^2 + 8.$$

This factors as $(\lambda^2+4)(\lambda^2+2)$, so the eigenvalues are the pure-imaginary pairs $\lambda = \pm 2i$ and $\lambda = \pm i\sqrt{2}$.

We compute the eigenvectors for $\lambda_1 = -2i$ first. Row reducing $A + 2iI$,

$$A + 2iI = \begin{bmatrix} 2i & 0 & 1 & 0 \\ 0 & 2i & 0 & 1 \\ -3 & 1 & 2i & 0 \\ 1 & -3 & 0 & 2i \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -i & 0 \\ 0 & 2 & 0 & -i \\ 0 & 1 & \frac{1}{2}i & 0 \\ 0 & -3 & 0 & \frac{3}{2}i \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & 0 & -i & 0 \\ 0 & 2 & 0 & -i \\ 0 & 0 & \frac{1}{2}i & \frac{1}{2}i \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & i \\ 0 & 2 & 0 & -i \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $2a + id = 0$, $2b - id = 0$, and $c + d = 0$; taking $d = -2$, $a = i$, $b = -i$, and $c = 2$, so a choice for the eigenvector is $\mathbf{v}_1 = \begin{bmatrix} i & -i & 2 & -2 \end{bmatrix}^T$. Hence, the complex solution is

$$e^{-2it}\mathbf{v}_1 = (\cos 2t - i \sin 2t) \begin{bmatrix} i \\ -i \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} \sin 2t + i \cos 2t \\ -\sin 2t - i \cos 2t \\ 2\cos 2t - 2i \sin 2t \\ -2\cos 2t + 2i \sin 2t \end{bmatrix},$$

so taking its real and imaginary parts gives two real solutions,

$$\mathbf{x}_1(t) = \begin{bmatrix} \sin 2t \\ -\sin 2t \\ 2\cos 2t \\ -2\cos 2t \end{bmatrix}, \qquad \mathbf{x}_2(t) = \begin{bmatrix} \cos 2t \\ -\cos 2t \\ -2\sin 2t \\ 2\sin 2t \end{bmatrix}.$$

We repeat this process for $\lambda_2 = -i\sqrt{2}$:

$$A + i\sqrt{2}I = \begin{bmatrix} i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{2} & 0 & 1 \\ -3 & 1 & i\sqrt{2} & 0 \\ 1 & -3 & 0 & i\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -i\sqrt{2} & 0 \\ 0 & 2 & 0 & -i\sqrt{2} \\ 0 & 0 & -i\frac{\sqrt{2}}{2} & i\frac{\sqrt{2}}{2} \\ 0 & 0 & i\frac{\sqrt{2}}{2} & -i\frac{\sqrt{2}}{2} \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & 0 & -i\sqrt{2} & 0 \\ 0 & 2 & 0 & -i\sqrt{2} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & -i\sqrt{2} \\ 0 & 2 & 0 & -i\sqrt{2} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \sqrt{2} & 0 & 0 & -i \\ 0 & \sqrt{2} & 0 & -i \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $\sqrt{2}a = id$, $\sqrt{2}b = id$, and $c = d$, so taking $d = \sqrt{2}$, we have the eigenvector $\mathbf{v}_2 = \begin{bmatrix} i & i & \sqrt{2} & \sqrt{2} \end{bmatrix}^T$, and hence the complex solution

$$e^{-i\sqrt{2}t}\mathbf{v}_1 = (\cos\sqrt{2}t - i \sin\sqrt{2}t) \begin{bmatrix} i \\ i \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} \sin\sqrt{2}t + i\cos\sqrt{2}t \\ \sin\sqrt{2}t + i\cos\sqrt{2}t \\ \sqrt{2}\cos 2t - i\sqrt{2}\sin 2t \\ \sqrt{2}\cos 2t - i\sqrt{2}\sin 2t \end{bmatrix}.$$

Its real and imaginary parts give the remaining two real solutions:

$$\mathbf{x}_3(t) = \begin{bmatrix} \sin\sqrt{2}t \\ \sin\sqrt{2}t \\ \sqrt{2}\cos 2t \\ \sqrt{2}\cos 2t \end{bmatrix}, \qquad \mathbf{x}_4(t) = \begin{bmatrix} \cos\sqrt{2}t \\ \cos\sqrt{2}t \\ -\sqrt{2}\sin 2t \\ -\sqrt{2}\sin 2t \end{bmatrix}.$$

The general solution is then

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \sin 2t \\ -\sin 2t \\ 2\cos 2t \\ -2\cos 2t \end{bmatrix} + c_2 \begin{bmatrix} \cos 2t \\ -\cos 2t \\ -2\sin 2t \\ 2\sin 2t \end{bmatrix} + c_3 \begin{bmatrix} \sin\sqrt{2}t \\ \sin\sqrt{2}t \\ \sqrt{2}\cos 2t \\ \sqrt{2}\cos 2t \end{bmatrix} + c_4 \begin{bmatrix} \cos\sqrt{2}t \\ \cos\sqrt{2}t \\ -\sqrt{2}\sin 2t \\ -\sqrt{2}\sin 2t \end{bmatrix}.$$

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Solution (b): The initial condition is $\mathbf{x}_0 = \begin{bmatrix} 3 & 1 & 0 & 0 \end{bmatrix}^T$, which we equate to $\mathbf{x}(0)$:

$$c_1 \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solving this linear system, $c_1 = c_3 = 0$, $c_2 = 1$, and $c_4 = 2$, so

$$\mathbf{x}(t) = \begin{bmatrix} \cos 2t \\ -\cos 2t \\ -2\sin 2t \\ 2\sin 2t \end{bmatrix} + 2 \begin{bmatrix} \cos\sqrt{2}t \\ \cos\sqrt{2}t \\ -\sqrt{2}\sin 2t \\ -\sqrt{2}\sin 2t \end{bmatrix}$$

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Solution (c): Writing the $x_i(t)$ components from above in their own vector, we have

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2 \begin{bmatrix} \cos\sqrt{2}t \\ \cos\sqrt{2}t \end{bmatrix} + \begin{bmatrix} \cos 2t \\ -\cos 2t \end{bmatrix} = \cos\sqrt{2}t \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \cos 2t \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This expresses the displacements as the sum of two different vibrational modes: the first term represents the motion of the two masses in the same direction at the lower frequency $\sqrt{2}$ rad/s, and the second their motion in opposite directions at the higher frequency 2 rad/s. ■