Math 19: Calculus **Summer 2010**

Practice Final Exam: Winter 2007 – Solutions

1. (*40 points*) Find $\frac{dy}{dx}$ for each function. Each answer should be a function of x only.

(a) (*10 points*) $y = \frac{2}{x-1} - \frac{1}{\sqrt{x}}$.

Solution: Write $y = 2(x-1)^{-1} - x^{-1/2}$. Then

$$\frac{dy}{dx} = -2(x-1)^{-2} + \frac{1}{2}x^{-3/2} = \boxed{-\frac{2}{(x-1)^2} + \frac{1}{2x^{3/2}}}$$

(b) (*10 points*) $y = (\sin x)^{\cos x}$.

Solution: Since $y = (\sin x)^{\cos x}$, $\ln y = \cos x \ln(\sin x)$. Then

$$\frac{1}{y} \cdot \frac{dy}{dx} = -\sin x \ln(\sin x) + \cos x \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = y\left(\frac{\cos^2 x}{\sin x} - \sin x \ln(\sin x)\right) = \boxed{(\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x \ln(\sin x)\right)}$$

(c) (*10 points*) $y = \sqrt{\tan(x^2)}$.

Solution: Since $y = (\tan x^2)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan x^2}}(\sec^2 x^2)(2x) = \boxed{\frac{x \sec^2 x^2}{\sqrt{\tan x^2}}}$$

(d) (*10 points*) $y = \frac{(2x+1)^4 \sin(x^2)}{(\ln x)\sqrt{3x-1}}$.

Solution: Since this is a complicated product of powers, we consider $\ln y$:

$$\begin{aligned}\ln y &= \ln\left(\frac{(2x+1)^4 \sin(x^2)}{(\ln x)\sqrt{3x-1}}\right)\\ &= 4\ln(2x+1) + \ln \sin x^2 - \ln(\ln x) - \frac{1}{2}\ln(3x-1)\\ \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{4(2)}{2x+1} + \frac{(\cos x^2)(2x)}{\sin x^2} - \frac{1/x}{\ln x} - \frac{1}{2}\frac{3}{3x-1}\\ \frac{dy}{dx} &= \boxed{\frac{(2x+1)^4 \sin(x^2)}{(\ln x)\sqrt{3x-1}}\left(\frac{8}{2x+1} + 2x\cot x^2 - \frac{1}{x\ln x} - \frac{3}{2(3x-1)}\right)}\end{aligned}$$

2. (*10 points*) Find the equation of the tangent line to the curve

$$e^{x^2} + e^{y^2} = 2e$$

at the point $(-1, 1)$.
Solution: Differentiate $e^{x^2} + e^{y^2} = 2e$ implicitly:

$$2xe^{x^2} + 2ye^{y^2}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2ye^{y^2}}{2xe^{x^2}} = -\frac{ye^{y^2}}{xe^{x^2}}.$$

At $(x, y) = (-1, 1)$,

$$-\frac{ye^{y^2}}{xe^{x^2}} = -\frac{(-1)e^{(-1)^2}}{(1)e^{1^2}} = \frac{e}{e} = 1,$$

so the tangent line is $y - 1 = 1(x + 1)$, or $\boxed{y = x + 2.}$

3. (*20 points*) Let

$$f(x) = \ln(x^2 - 1).$$

(a) (*10 points*) You must show all your work, but please write your final answers in the box.
Solution:

The domain of $f(x)$ is:	$(-\infty, -1) \cup (1, \infty)$
$f(x)$ is increasing on:	$(1, \infty)$
$f(x)$ is decreasing on:	$(-\infty, -1)$
$f(x)$ has local maxima at:	None
$f(x)$ has local minima at:	None
$f(x)$ is concave up on:	None
$f(x)$ is concave down on:	$(-\infty, -1) \cup (1, \infty)$

Since $f(x) = \ln(x^2 - 1)$, f is defined when $x^2 > 1$, so when $x > 1$ or when $x < -1$. Computing $f'(x)$ yields

$$f'(x) = \frac{2x}{x^2 - 1},$$

again defined for $|x| > 1$. Since $x^2 - 1 > 0$ for all such x, the sign of $f'(x)$ is the same as that of x. Hence, f is increasing for $x > 1$, and f is decreasing for $x < -1$.
Since $f'(x)$ is defined on the same domain as $f(x)$ is, and since $f'(x) = 0$ only when $x = 0$, f has no critical numbers, and hence no local minima or maxima.

Finally, we compute $f''(x)$:

$$f''(x) = \frac{(x^2-1)(2)-(2x)(2x)}{(x^2-1)^2} = \frac{-2(x^2+1)}{(x^2-1)^2}$$

For $|x| > 1$, $f''(x) < 0$, so f is concave down on all of its domain.

(b) (*4 points*) Compute the following four limits.

$$\lim_{x\to\infty} \ln(x^2-1) = \qquad \lim_{x\to-\infty} \ln(x^2-1) =$$

$$\lim_{x\to1^+} \ln(x^2-1) = \qquad \lim_{x\to-1^-} \ln(x^2-1) =$$

Solution: As $x \to \infty$ or $x \to -\infty$, $x^2-1 \to \infty$, so

$$\lim_{x\to\infty} \ln(x^2-1) = \lim_{x\to-\infty} \ln(x^2-1) = \infty.$$

As $x \to 1^+$ or $x \to -1^-$, $x^2-1 \to 0^+$, so

$$\lim_{x\to1^+} \ln(x^2-1) = \lim_{x\to-1^-} \ln(x^2-1) = \lim_{u\to0^+} \ln u = -\infty.$$

(c) (*1 points*) List all vertical and horizontal asymptotes of $y = \ln(x^2-1)$.

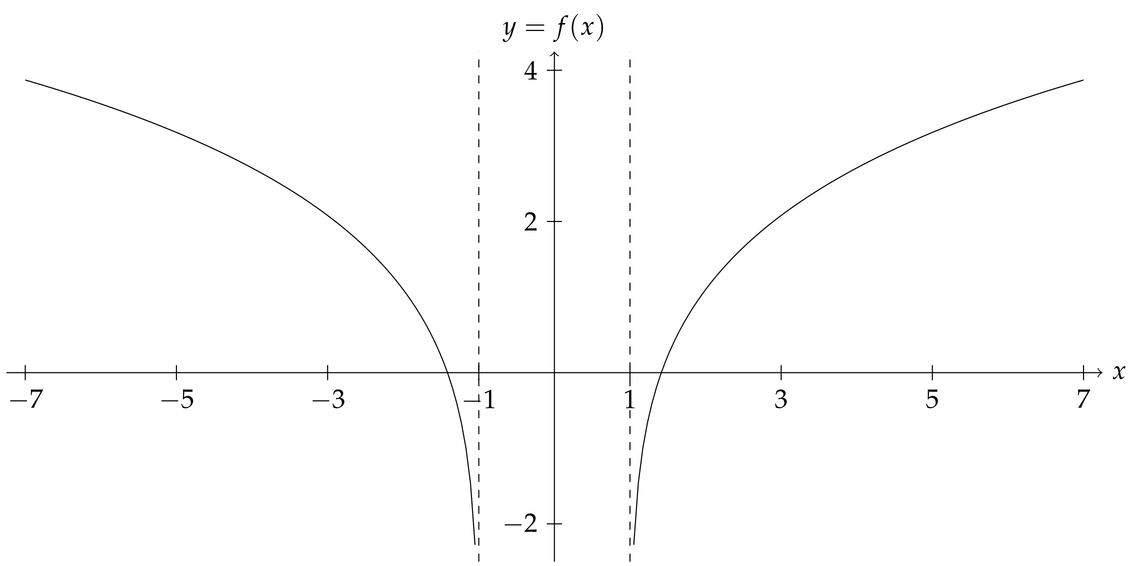
Solution: f has no horizontal asymptotes, and f has two vertical asymptotes, at $x = 1$ and $x = -1$.

(d) (*5 points*) Using your answers from parts (a) and (b), sketch a graph of

$$f(x) = \ln(x^2-1).$$

Even if your answers in parts (a) and (b) are wrong, if your sketch correctly uses those answers, you may earn partial credit.

Solution:



4. (*20 points*) A particle is moving along the curve $x^2 - 4xy - y^2 = -11$. Given that the x-coordinate of the particle is changing at 3 units/second, *how fast is the distance from the particle to the origin changing* when the particle is at the point $(1, 2)$? Hint: As an intermediate step, you should compute the value of $\frac{dy}{dt}$ when $x = 1$ and $y = 2$.

Solution: The distance z to the origin is given by $z^2 = x^2 + y^2$, so, differentiating with respect to the time t,

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

Differentiating the curve equation $x^2 - 4xy - y^2 = -11$ with respect to t,

$$2x\frac{dx}{dt} - 4y\frac{dx}{dt} - 4x\frac{dy}{dt} - 2y\frac{dy}{dt} = 0 \quad \Rightarrow \quad (x - 2y)\frac{dx}{dt} - (2x + y)\frac{dy}{dt} = 0.$$

At $(x, y) = (1, 2)$, $\frac{dx}{dt} = 3$, so $(-3)(3) - (4)\frac{dy}{dt}$, and $\frac{dy}{dt} = -\frac{9}{4}$. Also, at this point, $z = \sqrt{1^2 + 2^2} = \sqrt{5}$, so

$$\frac{dz}{dt} = \frac{1}{\sqrt{5}}\left((1)(3) + (2)\left(-\frac{9}{4}\right)\right) = \boxed{-\frac{3}{2\sqrt{5}}}.$$

5. (*20 points*) A balloon is rising at a constant speed of 1 m/sec. A girl is cycling along a straight road at a speed of 2 m/sec. When she passes under the balloon it is 3 m above her. How fast is the distance between the girl and the balloon increasing 2 seconds later?

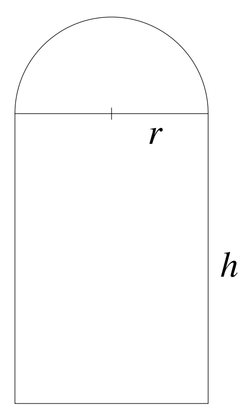
Solution: Let x represent the horizontal position of the girl from the balloon, and let y represent the height of the balloon. Then the distance z between the girl and the balloon is given by $z^2 = x^2 + y^2$. Differentiating with respect to time t,

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

At this time, $x = (2)(2) = 4$ and $y = 3 + (2)(1) = 5$, so $z = \sqrt{5^2 + 4^2} = \sqrt{41}$. Since $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 1$,

$$\frac{dz}{dt} = \frac{1}{\sqrt{41}}((4)(2) + (5)(1)) = \boxed{\frac{13}{\sqrt{41}}\,\frac{\text{m}}{\text{sec}}}.$$

6. (*20 points*) A Norman window consists of a rectangle surmounted by a semicircle, as shown. Given that the total area of the window is $A = 8 + 2\pi$, find the minimum possible perimeter of the window. (Please note the horizontal line between the rectangle and the semicircle does not count as part of the perimeter.) *Hint*: The total area has been carefully chosen so that the minimum perimeter occurs at a very simple value of r. If your optimal value of r is complicated, you have done something incorrectly.



Solution: Let A denote the area of the window, and P the perimeter of the window. Then

$$A = \frac{\pi}{2}r^2 + 2rh, \quad P = \pi r + 2r + 2h.$$

We seek to minimize P. Since the area is fixed at $8 + 2\pi$, we have

$$8 + 2\pi = \frac{\pi}{2}r^2 + 2rh \quad \Rightarrow \quad h = \frac{8 + 2\pi - \frac{\pi}{2}r^2}{2r} = \frac{4+\pi}{r} - \frac{\pi r}{4}.$$

Then

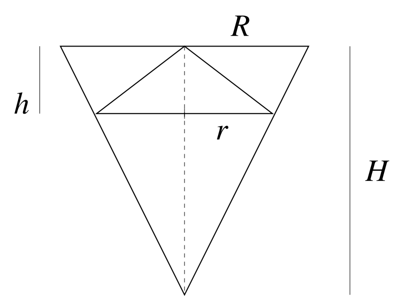
$$P = \pi r + 2r + 2\left(\frac{4+\pi}{r} - \frac{\pi r}{4}\right) = \frac{\pi}{2}r + 2r + \frac{2(\pi+4)}{r},$$

so $P'(r) = \frac{\pi}{2} + 2 - \frac{2(\pi+4)}{r^2}$. Setting this equal to 0, we get that

$$\frac{\pi+4}{2} = \frac{2(\pi+4)}{r^2} \quad \Rightarrow \quad r^2 = 4 \quad \Rightarrow \quad r = 2.$$

We examine the sign of $P'(r)$ for r on either side of $r = 2$. For $0 < r < 2$, $P'(r) < 0$, and for $r > 2$, $P'(r) > 0$, so by the First Derivative Test for Absolute Extreme, P has an absolute minimum at $r = 2$. At this value, $\boxed{P(2) = 2\pi + 8.}$

7. (*20 points*) Suppose you have a cone with *constant* height H and *constant* radius R, and you want to put a smaller cone "upside down" inside the larger cone (see figure). If h is the height of the smaller cone, what should h be to maximize the volume of the smaller cone? The optimal value of h will depend on H. Recall that the volume of a cone with base radius r and height h is given by the formula $V = \frac{1}{3}\pi r^2 h$.



Solution: Using the symbols in the diagram, the volume of the small cone is $\frac{\pi}{3}r^2 h$. By the similar triangles in the diagram, we relate r and h by

$$\frac{r}{R} = \frac{H-h}{H},$$

or $r = \frac{R}{H}(H-h)$. Hence, the volume is

$$V(h) = \frac{\pi}{3}\left(\frac{R}{H}(H-h)\right)^2 h = \frac{\pi R^2}{3H^2}(H^2 h - 2Hh^2 + h^3).$$

Hence, $V'(h) = \frac{\pi R^2}{3H^2}(H^2 - 4Hh + 3h^2)$. Setting this equal to 0 and factoring, we have that $(H - 3h)(H - h) = 0$, so $h = \frac{1}{3}H$ and $h = H$ are the critical points.

The domain of $V(h)$ is the closed interval $[0, H]$. Checking the endpoints, we see that $V(0) = 0$ and $V(H) = 0$, so the positive value

$$V\left(\frac{1}{3}H\right) = \frac{\pi R^2}{3H^2}\left(H - \frac{1}{3}H\right)^2\left(\frac{1}{3}H\right) = \frac{4\pi}{81}R^2 H$$

is the absolute maximum. Therefore, the height $\boxed{h = \frac{1}{3}H}$ gives a maximum volume.

8. (*10 points*) For parts (a) and (b), compute the given limits, if they exist. If you assert that a limit does not exist, you need to justify your answer to get full credit.

(a) (*5 points*) $\lim_{x\to\infty}\left(\sqrt{x^2-3x+1}-\sqrt{x^2+2}\right)$

Solution: We rewrite using the conjugate radical:

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2-3x+1}-\sqrt{x^2+2}\right) &= \lim_{x\to\infty}\left(\sqrt{x^2-3x+1}-\sqrt{x^2+2}\right)\frac{\sqrt{x^2-3x+1}+\sqrt{x^2+2}}{\sqrt{x^2-3x+1}+\sqrt{x^2+2}} \\
&= \lim_{x\to\infty}\frac{x^2-3x+1-(x^2+2)}{\sqrt{x^2-3x+1}+\sqrt{x^2+2}} \\
&= \lim_{x\to\infty}\frac{-3x-1}{\sqrt{x^2-3x+1}+\sqrt{x^2+2}} \\
&= \lim_{x\to\infty}\frac{-3x-1}{\sqrt{x^2-3x+1}+\sqrt{x^2+2}}\cdot\frac{1/x}{1/x} \\
&= \lim_{x\to\infty}\frac{-3-\frac{1}{x}}{\sqrt{1-\frac{3}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{2}{x^2}}} = \frac{-3}{1+1} = \boxed{-\frac{3}{2}}
\end{aligned}$$

(b) (*5 points*) $\lim_{x\to 2} e^{\frac{1}{x-2}}$

Solution: Let $u = \dfrac{1}{x-2}$. As $x \to 2^+$, $u \to \frac{1}{0^+} = +\infty$, and as $x \to 2^-$, $u \to \frac{1}{0^-} = -\infty$. Hence,

$$\begin{aligned}
\lim_{x\to 2^+} e^{\frac{1}{x-2}} &= \lim_{u\to+\infty} e^u = +\infty \\
\lim_{x\to 2^-} e^{\frac{1}{x-2}} &= \lim_{u\to-\infty} e^u = 0.
\end{aligned}$$

Since these values differ, the (two-sided) limit $\boxed{\text{does not exist.}}$