Math 19: Calculus **Summer 2010**

Practice Midterm Problems – Solutions

1. Circle "True" or "False." No explanation is needed.

(a) **True** **False** $f(x) = |x-2|$ is one-to-one.

(b) **True** **False** $\lim_{x\to 5}\left(\frac{2x}{x-5} - \frac{10}{x-5}\right) = \lim_{x\to 5}\frac{2x}{x-5} - \lim_{x\to 5}\frac{10}{x-5}$

(c) **True** **False** A function can have infinitely many horizontal asymptotes.

(d) **True** **False** If f is continuous on $[0,2]$, then f is differentiable on $[0,2]$.

(e) **True** **False** The nth derivative of $f(x) = e^{2x}$ is $2^n e^{2x}$.

Solution:

(a) False: Since $f(1) = f(3) = 1$, f is not one-to-one.

(b) False: Neither $\lim_{x\to 5}\frac{2x}{x-5}$ nor $\lim_{x\to 5}\frac{10}{x-5}$ exists, so their difference does not make sense. However,

$$\lim_{x\to 5}\left(\frac{2x}{x-5} - \frac{10}{x-5}\right) = \lim_{x\to 5}\frac{2x-10}{x-5} = \lim_{x\to 5} 2 = 2.$$

(c) False: A function can have at most two horizontal asymptotes, corresponding to the finite limit values of the function as $x \to +\infty$ and $x \to -\infty$.

(d) False: Continuity does not imply differentiability; for example, $|x|$ is continuous but not differentiable at $x = 0$.

(e) True: Since $\frac{d}{dx}(e^{2x}) = 2e^{2x}$, differentiating n times produces n factors of 2, and hence the 2^n coefficient.

2. The graph of $f(x)$ is shown. Answer the following questions and explain your reasoning:

(a) What is the domain of f?
Solution: The domain of f is $[-2,2]$.

(b) What is the range of f?
Solution: The range of f is $[-1,1]$.

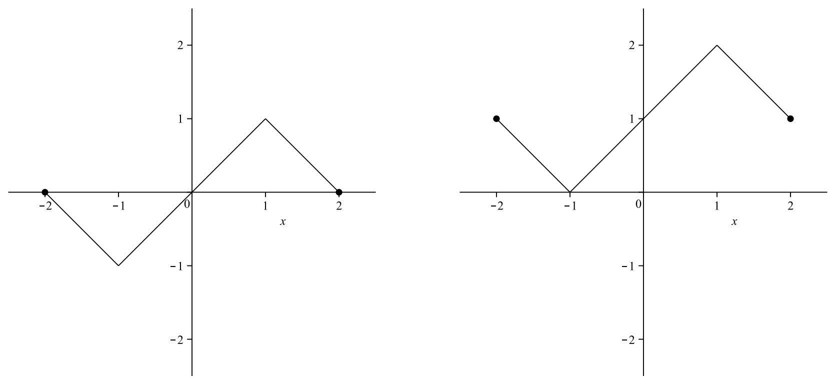
(c) Is f one-to-one?
Solution: f is not a one-to-one function, as it fails the horizontal line test for each y between -1 and 1.

(d) Where is f not differentiable?

Solution: f is not differentiable at $x = -1$ and at $x = 1$, as it has a "corner" at each x-value.

(e) Sketch the graph of $-f(-x) + 1$ on the coordinate system.

Solution:



3. For each of the following limits, evaluate it or show it does not exist.

(a) $\lim_{x\to -1} \dfrac{x^2 - 3x - 4}{x+1}$

Solution: Substituting in $x = -1$ gives $0/0$, so we must rewrite the limit:

$$\lim_{x\to -1} \frac{x^2 - 3x - 4}{x+1} = \lim_{x\to -1} \frac{(x+1)(x-4)}{x+1} = \lim_{x\to -1} (x-4) = (-1-4) = -5.$$

(b) $\lim_{x\to \frac{1}{2}} \ln(\sin(\pi x))$

Solution: Since ln and sin are both continuous functions where they are defined, we can substitute in $x = 1/2$:

$$\lim_{x\to \frac{1}{2}} \ln(\sin(\pi x)) = \ln(\sin(\pi \frac{1}{2})) = \ln(1) = 0.$$

(c) $\lim_{x\to 2} (x^2 - 4)^2 \sin\left(\dfrac{1}{x-2}\right)$

Solution: Note that for all $x \neq 2$,

$$-1 \leq \sin\left(\frac{1}{x-2}\right) \leq 1.$$

Since $(x^2-4)^2 \geq 0$ for all x,

$$-(x^2-4)^2 \leq (x^2-4)^2 \sin\left(\frac{1}{x-2}\right) \leq (x^2-4)^2$$

Since $\lim_{x\to 2}(x^2-4)^2 = 0$, the limits of the left and right functions as $x \to 2$ are both 0. By the Squeeze Theorem, then,

$$\lim_{x\to 2}(x^2-4)^2 \sin\left(\frac{1}{x-2}\right) = 0.$$

(d) $\displaystyle\lim_{x\to\infty} \frac{3-x}{x^2-3x+2}$

Solution: Substituting in $x = +\infty$ gives $-\infty/(\infty-\infty)$, so we must rewrite the limit. We divide the numerator and denominator by x^2, the highest power of x appearing in the denominator:

$$\lim_{x\to\infty} \frac{3-x}{x^2-3x+2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{3}{x^2}-\frac{1}{x^2}}{1-\frac{3}{x}+\frac{2}{x^2}} = \frac{0-0}{1-0+0} = \frac{0}{1} = 0.$$

(e) $\displaystyle\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} e^x & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ \tan^2 x + 1 & \text{if } x > 0 \end{cases}$

Solution: Since $f(x)$ has a piecewise definition that changed at $x = 0$, we evaluatue the left- and right-hand limits separately:

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} e^x = e^0 = 1, \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^-} \tan^2 x + 1 = \tan^2 0 + 1 = 1. \end{aligned}$$

Since these one-sided limits exist and agree, $\lim_{x\to 1} f(x) = 1$.

4. Let $g(t) = \dfrac{t+3}{t-1}$.

(a) Find the equation(s) of all vertical asymptote(s) of g.

Solution: We look for where $g(t)$ could have an infinite discontinuity. Since $t+3$ and $t-1$ are both continuous, a discontinuity could occur only when the denominator $t-1$ is 0, so at $t = 1$. At this t-value, $t+3 = 4 \neq 0$, so $g(t)$ does indeed "blow up" at $t = 1$. Hence, $t = 1$ is the only vertical asymptote of g.

(b) Find the equation(s) of all horizontal asymptotes of g.

Solution: We note that attempting to evaluate g at ∞ yields ∞/∞, so we must rewrite the expression in the limit. We divide the numerator and denominator by the highest power of t in the denominator:

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \frac{t+3}{t-1} \cdot \frac{\frac{1}{t}}{\frac{1}{t}} = \lim_{t\to\infty} \frac{1+\frac{3}{t}}{1-\frac{1}{t}} = \frac{1+0}{1-0} = 1.$$

Similarly, $\lim_{t\to-\infty} g(t) = 1$, so $y = 1$ is the only horizontal asymptote of g.

Solution: (Alternate) We first rewrite $g(t)$ as a polynomial plus a simpler rational function:

$$g(t) = \frac{t+3}{t-1} = \frac{(t-1)+4}{t-1} = 1 + \frac{4}{t-1}.$$

Then

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} 1 + \frac{4}{t-1} = 1 + 0 = 1,$$

and similarly $\lim_{t\to-\infty} g(t) = 1$, so $y = 1$ is the only horizontal asymptote of g.

(c) Find $g^{-1}(t)$.

Solution: We use the form of $g(t)$ from the alternate solution to part (b). Then

$$y = g(t) = 1 + \frac{4}{t-1},$$

so $y - 1 = \frac{4}{t-1}$. Then $t - 1 = \frac{4}{y-1}$, so

$$t = 1 + \frac{4}{y-1} = \frac{y-1+4}{y-1} = \frac{y+3}{y-1} = g^{-1}(y).$$

Substituting t for y, $g^{-1}(t) = \frac{t+3}{t-1}$.

5. (a) Let $f(x) = x^2 - \sin x$. Compute $f'(x)$.

Solution: We compute that $f'(x) = 2x - \cos x$.

(b) Show there exists a number a between $[0, \frac{\pi}{2}]$ such that the graph of $x^2 - \sin x$ has a horizontal tangent line at a.

Solution: We must show that, for some a in $(0, \frac{\pi}{2})$, $f'(a) = 0$, We note that $f'(0) = 2(0) - \cos(0) = -1 < 0$ and $f'(\pi/2) = 2(\frac{\pi}{2}) - \cos(\frac{\pi}{2}) = \pi - 0 = \pi > 0$. Since $f'(x)$ is a continuous function, the Intermediate Value theorem states that $f'(a) = 0$ for some a in $(0, \frac{\pi}{2})$.

6. (a) Using the limit definition of the derivative, compute the derivative of $f(x) = 2\sqrt{x}$.

Solution: We use the limit definition of $f'(x)$:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{2\sqrt{x+h} - 2\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{2(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h\to 0} \frac{2((x+h) - x)}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{2h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{2}{(\sqrt{x+h} + \sqrt{x})} = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}} \end{aligned}$$

(b) Find the equation of the tangent line to the curve when $x = 1$.

Solution: At $x = 1$, $f(1) = 2\sqrt{1} = 2$, and $f'(1) = \frac{1}{\sqrt{1}} = 1$. Then the equation of the tangent line is

$$y - 2 = 1(x - 1) = x - 1,$$

or $y = x + 1$.

7. Find the derivatives of the following functions:

(a) $f(x) = x^5 - x^{3/4} + 1$

Solution: By the Power Rule, $f'(x) = 5x^4 - \frac{3}{4}x^{-1/4}$.

(b) $f(x) = x \ln x$

Solution: Using the product rule,

$$f'(x) = (x)' \ln x + x(\ln x)' = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1.$$

(c) $f(x) = \sin(2e^x)$

Solution: Write $f = h \circ g$, where $h(u) = \sin u$ and $u = g(x) = 2e^x$. Then

$$f'(x) = h'(g(x))g'(x) = \cos(2e^x)(2e^x) = 2e^x \cos(2e^x).$$

(d) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

Solution: We use the quotient rule:

$$f'(x) = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)} = \frac{4x}{(x^2+1)^2}.$$

(e) $f(x) = \ln\left(\frac{\sqrt{x}\cot x}{e^x}\right)$

Solution: We first write $f(x) = g(h(x))$, where $g(u) = \ln u$ and $h(x) = \frac{\sqrt{x}\cot x}{e^x}$. Using the product and quotient rules, we compute

$$\begin{aligned} h'(x) &= \frac{e^x(\frac{1}{2\sqrt{x}}\cot x + \sqrt{x}(-\csc^2 x)) - (e^x)(\sqrt{x}\cot x)}{(e^x)^2} \\ &= \frac{\cot x - 2x\csc^2 x - 2x\cot x}{2\sqrt{x}e^x}. \end{aligned}$$

Since $g'(u) = 1/u$,

$$\begin{aligned} f'(x) &= \frac{e^x}{\sqrt{x}\cot x} \cdot \frac{\cot x - 2x\csc^2 x - 2x\cot x}{2\sqrt{x}e^x} \\ &= \frac{\cot x - 2x\csc^2 x - 2x\cot x}{2x\cot x} = \frac{1}{2x} - \frac{\csc^2 x}{\cot x} - 1. \end{aligned}$$

Alternately, $f(x) = \ln(\sqrt{x}) + \ln(\cot x) - \ln(e^x) = \frac{1}{2}\ln x + \ln(\cot x) - x$, so

$$f'(x) = \frac{1}{2x} + \frac{-\csc^2 x}{\cot x} - 1 = \frac{1}{2x} - \frac{\csc^2 x}{\cot x} - 1.$$

(f) $f(x) = |x|$

Solution: Recall that $f(x) = |x|$ is defined piecewise by $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$. For $x > 0$, $f(x) = x$, so $f'(x) = 1$. For $x < 0$, $f(x) = -x$, so $f'(x) = -1$. For $x = 0$, however, we compute the left- and right-hand limits of the difference quotient:

$$\begin{aligned} \lim_{x\to 0^+} \frac{|x| - 0}{x - 0} &= \lim_{x\to 0^+} \frac{x}{x} = 1, \\ \lim_{x\to 0^-} \frac{|x| - 0}{x - 0} &= \lim_{x\to 0^-} \frac{-x}{x} = -1. \end{aligned}$$

Since these limits exist but do not agree, $f'(0)$ does not exist. Hence, $f'(x)$ is

$$f'(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

8. The displacement (in centimeters) of a particle moving back and forth along a straight line is given by $s(t) = 2^t + t^3 + 1$, where t is measured in seconds.

(a) Find the average velocity of the particle from $t = 1$ to $t = 3$.

Solution: We compute the average velocity, in centimeters per second:

$$v_{avg} = \frac{s(3) - s(1)}{3 - 1} = \frac{(2^3 + 3^3 + 1) - (2^1 + 1 + 1)}{2} = \frac{32}{2} = 16.$$

(b) Find the instantaneous velocity of the particle at $t = 1$.

Solution: We compute $s'(t)$:

$$s'(t) = (\ln 2)2^t + 3t^2.$$

Then $s'(1) = (\ln 2)2^1 + 3(1)^2 = 2\ln 2 + 3$, in cm/s.

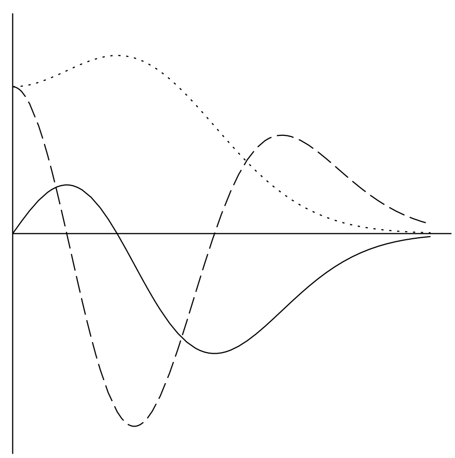
(c) Find the acceleration of the particle at $t = 1$.

Solution: We compute $s''(t)$ by differentiating $s'(t)$ again:

$$s''(t) = (\ln 2)^2 2^t + 6t$$

Then $s''(1) = 2(\ln 2)^2 + 6$, in cm/s^2.

9. The figure shows the graphs of f, f', and f''. Identify each curve and explain your choices.



Solution: None of the graphs is 0 when the dashed graph has its right-most horizontal tangent line, so the dashed-graph function is the highest derivative present and therefore must be f''. The dashed graph has height 0 precisely when the solid graph has a horizontal tangent line, so it is the derivative of the solid-graph function. Hence, the solid graph represents the graph of f'. Finally, the solid graph has height 0 exactly when the dotted graph has a horizontal tangent, so this dotted graph represents the graph of f.

10. Sketch a possible graph of $f(x)$ which satisfies all of the following conditions:

(i) $f(0) = 1$

(ii) $\lim_{x \to -\infty} f(x) = 0$

(iii) $f'(0) = 1$

(iv) f is increasing on $[-1, 1]$

(v) $\lim_{x \to 3^-} f(x) = 5$

(vi) $\lim_{x \to 3^+} f(x) = 2$

(vii) f is decreasing on $[3, \infty)$

(viii) $\lim_{x \to \infty} f(x) = -\infty$

Solution: Here is one possible graph of such a function f: