Math 19: Calculus **Summer 2010**

Midterm Exam — July 20, 2010, 7:00 to 9:00 PM

Name: Solution Key

Section (circle one): Eric · 1:15 PM Anca · 2:15 PM

- **You have a maximum of 2 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids are allowed.
- Read each question carefully. Show your work and justify your answers for full credit. You do not need to simplify your answers unless instructed to do so. You may use results from class, but if you cite a theorem you should check that the hypotheses are explicitly verified.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign to indicate that you have read and agree to the following statement:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the Stanford Honor Code with respect to this examination."

Signature: ____________________

Grading

1	/8	**6**	/10
2	/12	**7**	/10
3	/10	**8**	/10
4	/10	**9**	/10
5	/10	**10**	/10
		Total	/100

1. (*8 points*) Circle "True" or "False." No explanation is needed.

(a) **True** **[False]** If $f(x)$ is not defined at $x = 1$, then $\lim_{x \to 1} f(x)$ does not exist.

Solution: The definition of $\lim_{x \to 1} f(x)$ does not depend on the existence or value of $f(1)$.

(b) **[True]** **False** The nth derivative of e^x is e^x.

Solution: This immediately follows from the fact that the derivative of e^x is the function e^x itself.

(c) **True** **[False]** $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ does not exist because $\frac{0}{0}$ is an indeterminate form.

Solution: With an indeterminate form like $\frac{0}{0}$, we need to do some work to find the limit. In this case, we have

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4.$$

(d) **[True]** **False** The sum of two continuous functions is continuous.

Solution: This is a property of continuous functions; see Theorem 4 in Section 2.4 of the textbook.

(e) **[True]** **False** A function can have two different horizontal asymptotes.

Solution: The function $f(x) = \frac{\sqrt{x^2+1}}{x+1}$ has two different horizontal asymptotes, $y = 1$ and $y = -1$.

(f) **[True]** **False** If p is a polynomial function, then $\lim_{x \to a} p(x) = p(a)$.

Solution: This holds because polynomials are continuous.

(g) **True** **[False]** If f and g are differentiable, then $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$.

Solution: Consider $f(x) = x$ and $g(x) = x$. Then $\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[x^2] = 2x$ but $f'(x)g'(x) = 1 \cdot 1 = 1$, so $\frac{d}{dx}[f(x)g(x)] \neq f'(x)g'(x)$.

(h) **True** **[False]** $\frac{d}{dx}(\cos x) = \sin x$.

Solution: In fact $\frac{d}{dx}(\cos x) = -\sin x$ (a sign is missing above). All the "co" functions (cosine, cotangent, cosecant) produce a minus sign when we take the derivative.

2. (*12 points*) Evaluate the following limits, or show they do not exist:

(a) (*3 points*) $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right)$

Solution: Substituting $t = 0$ produces the indeterminate form $\infty - \infty$. We compute

$$\begin{aligned}
\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right) &= \lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}} && \left(\frac{0}{0}, \text{ so rewrite again}\right)\\
&= \lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}\cdot\frac{1+\sqrt{1+t}}{1+\sqrt{1+t}} && (\text{conjugate radical})\\
&= \lim_{t\to 0}\frac{1-(1+t)}{t\sqrt{1+t}\,(1+\sqrt{1+t})}\\
&= \lim_{t\to 0}\frac{-t}{t\sqrt{1+t}\,(1+\sqrt{1+t})}\\
&= \lim_{t\to 0}\frac{-1}{\sqrt{1+t}\,(1+\sqrt{1+t})}\\
&= -\frac{1}{2}.
\end{aligned}$$

Hence $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right) = \boxed{-\frac{1}{2}.}$

(b) (*3 points*) $\lim_{x\to 2}\frac{x^2-5x+6}{x^2+2x-8}$

Solution: We compute

$$\lim_{x\to 2}\frac{x^2-5x+6}{x^2+2x-8} = \lim_{x\to 2}\frac{(x-2)(x+3)}{(x-2)(x+4)} = \lim_{x\to 2}\frac{x+3}{x+4} = -\frac{1}{6}.$$

Hence $\lim_{x\to 2}\frac{x^2-5x+6}{x^2+2x-8} = \boxed{-\frac{1}{6}.}$

(c) (*3 points*) $\lim_{w \to 2\pi} \ln(\cos w)$

Solution: Since $\cos w$ is a continuous function, $\lim_{w \to 2\pi} \cos w = \cos(2\pi) = 1$. Since $\ln(x)$ is a continuous function, $\lim_{x \to 1} \ln(x) = \ln(1) = 0$. Putting this together, letting $x = \cos w$, we get

$$\lim_{w \to 2\pi} \ln(\cos w) = \lim_{x \to 1} \ln(x) = 0.$$

Hence $\lim_{w \to 2\pi} \ln(\cos w) = \boxed{0.}$

(d) (*3 points*) $\lim_{u \to \pi} \frac{\cos u}{(u - \pi)^2}$

Solution: Since $\cos w$ is a continuous function, $\lim_{u \to \pi} \cos w = \cos(\pi) = -1$. As $u \to \pi$, $(u - \pi)^2$ approaches 0 and remains positive. Putting this together, we get

$$\lim_{u \to \pi} \frac{\cos u}{(u - \pi)^2} = \frac{-1}{+0} = -\infty.$$

Hence $\lim_{u \to \pi} \frac{\cos u}{(u - \pi)^2} = \boxed{-\infty.}$

3. (*10 points*)

(a) (*4 points*) Complete the following definition: $f(x)$ is continuous at $x = a$ if

Solution:

1. $f(a)$ is defined,
2. $\lim_{x \to a} f(x)$ exists, and
3. $\lim_{x \to a} f(x) = f(a)$.

(b) (*6 points*) Find the value of c so that the function

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2, \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

is continuous for all real x.

Solution: First observe that f is continuous at $a < 2$ since $cx^2 + 2x$ is a polynomial and f is also continuous at $a > 2$ since $x^3 - cx$ is a polynomial. We need to find a c such that $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ so that f is continuous at 2. Since

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (cx^2 + 2x) = 4c + 4, \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (x^3 - cx) = 8 - 2c, \end{aligned}$$

we need $4c + 4 = 8 - 2c$. Solving for c,

$$4c + 4 = 8 - 2c \quad \Rightarrow \quad 6c = 4 \quad \Rightarrow \quad c = \frac{2}{3}.$$

Therefore $\boxed{c = \frac{2}{3}.}$

4. (*10 points*) Show that the equation $x + \ln x = \frac{2}{x}$ has a solution c in the interval $(1, 2)$.

Solution: Let

$$f(x) = x + \ln x - \frac{2}{x},$$

so that $f(c) = 0$ if and only if c is a solution of the equation above. Note that f is a continuous function on $[1, 2]$.
We want to find a $c \in (1, 2)$ such that $f(c) = 0$. We have

$$f(1) = 1 + \ln(1) - \frac{2}{1} = -1 < 0,$$
$$f(2) = 2 + \ln(2) - \frac{2}{2} = 1 + \ln(2) > 0.$$

Therefore, by the Intermediate Value Theorem, there exists a $c \in (1, 2)$ such that $f(c) = 0$.

5. (*10 points*) Let $f(x) = e^{1/x}$.

(a) (*3 points*) Find the equation(s) of all vertical asymptote(s), or explain why none exists. Justify your answer with limit calculations.

Solution: We look for infinite discontinuities of $f(x) = e^{1/x}$. We note that $f(x)$ is the composite of the functions e^u and $u = \dfrac{1}{x}$. e^u is continuous at all real numbers u, and $\frac{1}{x}$ is continuous except at $x = 0$, so the only possible place for a discontinuity to occur is at $x = 0$.

We check for an infinite discontinuity by evaluating the left- and right-hand limits of $f(x)$ as $x \to 0$. Since $\lim_{x\to 0^+} \dfrac{1}{x} = +\infty$ and $\lim_{x\to 0^-} \dfrac{1}{x} = -\infty$, we have

$$\lim_{x\to 0^+} e^{1/x} = \lim_{u\to +\infty} e^u = +\infty,$$
$$\lim_{x\to 0^-} e^{1/x} = \lim_{u\to -\infty} e^u = 0.$$

Hence, the left-hand limit is sufficient to tell us that $f(x)$ has an infinite discontinuity at $x = 0$. Thus, $\boxed{x = 0}$ is the only vertical asymptote of $y = f(x)$.

(b) (*3 points*) Find the equation(s) of all horizontal asymptote(s), or explain why none exists. Justify your answer with limit calculations.

Solution: We evaluate the limit of $f(x)$ as $x \to +\infty$ and as $x \to -\infty$. By the continuity of e^x, we have

$$\lim_{x\to +\infty} e^{1/x} = e^{\left(\lim_{x\to +\infty} \frac{1}{x}\right)} = e^0 = 1,$$
$$\lim_{x\to -\infty} e^{1/x} = e^{\left(\lim_{x\to -\infty} \frac{1}{x}\right)} = e^0 = 1.$$

Hence, $\boxed{y = 1}$ is the only horizontal asymptote of $y = f(x)$.

(c) (*2 points*) $f(x)$ is a one-to-one function. Find an expression for its inverse function, $f^{-1}(x)$, in terms of x.

Solution: We write $y = f(x) = e^{1/x}$ and then solve for x:

$$\begin{aligned} y &= e^{1/x} \\ \ln y &= \ln(e^{1/x}) = \frac{1}{x} \\ \frac{1}{\ln y} &= x \\ f^{-1}(y) = x &= \frac{1}{\ln y}. \end{aligned}$$

Substituting x for y gives $\boxed{f^{-1}(x) = \frac{1}{\ln x}.}$

(d) (*2 points*) Find the domain of $f^{-1}(x)$.

Solution: We compute the domain of $f^{-1}(x) = \frac{1}{\ln x}$. Since this is the composite of $u = \ln x$ and $y = \frac{1}{u}$, we require that $\ln x$ be defined and not equal to 0. The former condition gives $x > 0$, and the latter says we must exclude $\ln x = 0$, so $x = e^0 = 1$. Hence, the domain is $\boxed{x > 0 \text{ and } x \neq 1,}$ or $\boxed{(0,1) \cup (1,+\infty)}$ in interval notation.

6. (*10 points*)

(a) (*3 points*) State a definition of the derivative of $f(x)$ at $x = a$:

Solution: $\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$ or $\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.}$ Note that while we *interpret* $f'(a)$ to be the slope of the tangent line to $y = f(x)$ at $x = a$, this is not a definition of $f'(a)$.

(b) (*4 points*) Use your definition to compute the derivative of the function $f(x) = \dfrac{1}{x-2}$ at $x = 3$.

Solution: We use the "h" form of the limit definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\dfrac{1}{x+h-2} - \dfrac{1}{x-2}}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \cdot \frac{(x-2) - (x+h-2)}{(x+h-2)(x-2)} \\ &= \lim_{h \to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-2)(x-2)} = \lim_{h \to 0} \frac{-1}{(x+h-2)(x-2)} = \frac{-1}{(x-2)^2}. \end{aligned}$$

We now evaluate at $x = 3$: $f'(3) = \dfrac{-1}{(3-2)^2} = \dfrac{-1}{(1)^2} = \boxed{-1.}$

(c) (*3 points*) Find the tangent line to the graph $y = f(x)$ at the point $(3, 1)$.

Solution: The equation of the tangent line to $y = f(x)$ at $x = a$ is

$$y - f(a) = f'(a)(x - a).$$

Plugging in $a = 3$, $f(a) = 1$, and $f'(a) = -1$ from part (b), we have

$$\boxed{y - 1 = (-1)(x - 3)}$$

which can be simplified to $\boxed{y = 4 - x.}$

Note on grading: If an incorrect number was computed in part (b), full credit was still given if that number was used correctly as the slope in part (c).

7. (*10 points*) Compute the derivatives of the following functions. Show your work for full credit.

(a) (*3 points*) $f(x) = x^5 + x^{-3}e^x$

Solution: Applying the power rule and the product rule,

$$f'(x) = \boxed{5x^4 + (-3)x^{-4}e^x + x^{-3}e^x.}$$

(b) (*3 points*) $r = \dfrac{\sec\theta}{1+\sec\theta}$ (*Hint*: $\sec\theta = \dfrac{1}{\cos\theta}$.)

Solution: The derivative of $\sec\theta$ is $\sec\theta\tan\theta$, so by the quotient rule

$$\begin{aligned} r' &= \frac{(\sec\theta)'(1+\sec\theta) - \sec\theta(1+\sec\theta)'}{(1+\sec\theta)^2} \\ &= \frac{\sec\theta\tan\theta(1+\sec\theta) - \sec\theta(\sec\theta\tan\theta)}{(1+\sec\theta)^2} \\ &= \boxed{\frac{\sec\theta\tan\theta}{(1+\sec\theta)^2}.} \end{aligned}$$

Alternately, using the hint, we can rewrite r as follows:

$$\begin{aligned} r &= \frac{\frac{1}{\cos\theta}}{1+\frac{1}{\cos\theta}} = \frac{\frac{1}{\cos\theta}}{\frac{\cos\theta+1}{\cos\theta}} = \frac{1}{\cos\theta+1} \\ r' &= \left(\frac{1}{\cos\theta+1}\right) = (-1)\frac{-\sin\theta}{(\cos\theta+1)^2} = \boxed{\frac{\sin\theta}{(\cos\theta+1)^2}.} \end{aligned}$$

(c) (*4 points*) $y = (1+2w)^{10}$

Solution: We use the chain rule, where $g(w) = 1+2w$ and $f(w) = w^{10}$. Then $g'(w) = 2$, and $f(w) = 10w^9$. So

$$f'(g(w))g'(w) = 10(1+2w)^9 \cdot 2 = \boxed{20(1+2w)^9.}$$

8. (*10 points*) A particle moves along a horizontal line, with its position given by the function $s(t) = t^3 - 3t^2 + 2t$, where s is in centimeters and t is in seconds.

(a) (*2 points*) Find the average velocity of the particle from $t = 1$ to $t = 2$.

Solution: The average velocity is given by $\frac{\text{displacement}}{\text{time}}$, so it is equal to

$$\frac{s(2) - s(1)}{2 - 1} = \frac{0 - 0}{1} = \boxed{0.}$$

The units here are centimeters per second, but this information was not required.

(b) (*4 points*) Find the instantaneous velocity $v(t)$ of the particle.

Solution: The instantaneous velocity is the derivative of the position function:

$$v(t) = s'(t) = \boxed{3t^2 - 6t + 2.}$$

(c) (*4 points*) Find the acceleration $a(t)$ of the particle.

Solution: The acceleration is the second derivative of the position function, or the derivative of the instantaneous velocity:

$$a(t) = v'(t) = (3t^2 - 6t + 2)' = \boxed{6t - 6.}$$

9. (*10 points*) For what values of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ when $x = 2$?

Solution: When $x = 2$, the parabola and its tangent line at this point pass through the same point. In other words we must have $y_{\text{parabola}}(2) = y_{\text{line}}(2)$. But $y_{\text{parabola}}(2) = 4a$, and $y_{\text{line}}(2) = b - 4$; since these are equal, we have that

$$4a = b - 4.$$

We also know that the line $2x + y = b$ is tangent to the parabola at $x = 2$. So the slope of the line must be equal to the derivative of the function $y = ax^2$ at $x = 2$. Now, $y'(x) = 2ax$, so $y'(2) = 4a$. Rewriting the equation of the line $2x + y = b$ as $y = -2x + b$, we can see that the slope of the line is -2. Thus, we know that $y'(2) = 4a = -2$, so

$$\boxed{a = -\frac{1}{2}.}$$

Using the equation $4a = b - 4$ to solve for b, we get $\boxed{b = 2.}$

10. (*10 points*) On the coordinate axes below, sketch the graph of a function $f(x)$ with the following properties:

(i) $\lim_{x\to-\infty} = \infty$

(ii) $\lim_{x\to\infty} = 2$

(iii) $f(x)$ has a jump discontinuity at $x = -2$

(iv) $f(x)$ is continuous but not differentiable at $x = 4$

(v) $f(x)$ has a vertical asymptote at $x = 0$

(vi) $f(x)$ is continuous from the right at $x = 0$

(vii) $f'(1) = -1$

(viii) $f(2) = 1$

(ix) $f'(2) = 0$

(x) $f'(3) = 2$

Solution: Answers may vary, but here is a representative graph exhibiting all of the features: