A tank contains 10 liters of pure water. Salt water containing 20 grams of salt per liter is pumped into the tank at 2 liters per minute.

1. Express the salt concentration $C(t)$ after $t$ minutes (in g/L).
2. What is the long-term concentration of salt, i.e., $\lim_{t \to \infty} C(t)$?

Solution:

1. The concentration is, in units of g/L,

$$C(t) = \frac{\text{total salt}}{\text{total volume}} = \frac{20 \cdot 2 \cdot t}{10 + 2 \cdot t} = \frac{20t}{5 + t}$$

2. The long-term concentration is, in units of g/L,

$$\lim_{t \to +\infty} \frac{20t}{5 + t} = \lim_{t \to +\infty} \frac{20t}{5 + t} \cdot \frac{1/t}{1/t} = \lim_{t \to +\infty} \frac{20}{5/t + 1} = 20$$
Find the values of $a$ and $b$ that make $f(x)$ continuous for all real $x$.

$$f(x) = \begin{cases} 
be^x + a + 1, & x \leq 0 \\
ax^2 + b(x + 3), & 0 < x \leq 1 \\
a \cos(\pi x) + 7bx, & x > 1 
\end{cases}$$

**Solution:** We note that the functions are continuous on their domains, so we check that the left- and right-hand limits agree at the boundary $x$-values. At $x = 0$,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} be^x + a + 1 = b + a + 1,$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} ax^2 + b(x + 3) = 3b,$$

so $b + a + 1 = 3b$, and $a = 2b - 1$. Next, at $x = 1$,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} ax^2 + b(x + 3) = a + 4b,$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} a \cos(\pi x) + 7bx = -a + 7b,$$

so $a + 4b = -a + 7b$, and $2a = 3b$. Solving this linear system in $a$ and $b$ yields $b = 2$ and $a = 3$ as the only solution.
3 Sketch the graph of a function $f$ with the following properties:

- $\lim_{x \to 1} f(x) = 2$, but $f(1) = 1$
- $\lim_{x \to 3} f(x) = +\infty$
- $\lim_{x \to 2^+} f(x) = -1$, $\lim_{x \to 2^-} f(x) = 3$
- $\lim_{x \to +\infty} f(x) = -2$
- $\lim_{x \to -\infty} f(x) = -\infty$

Solution: Answers may vary, but here is a representative solution:
Show that the equation \( \sqrt{x - 5} = \frac{1}{x + 3} \) has at least one real solution.

**Solution:** Let \( f(x) = \sqrt{x - 5} - \frac{1}{x + 3} \), so that \( f(x) = 0 \) if and only if \( x \) is a solution to the equation. Then \( f \) is defined and continuous for all \( x \geq 5 \). Evaluating \( f \) at 5 and at 6, we see that

\[
\begin{align*}
  f(5) &= \sqrt{5 - 5} - \frac{1}{5 + 3} = -\frac{1}{8} < 0 \\
  f(6) &= \sqrt{6 - 5} - \frac{1}{6 + 3} = \frac{8}{9} > 0.
\end{align*}
\]

By the Intermediate Value Theorem, there is some \( c \) in the interval \((5, 6)\) so that \( f(c) = 0 \), so \( f \) has at least one root.

(In fact, it is possible to reduce this equation to the cubic polynomial equation \((x - 5)(x + 3)^2 - 1 = 0\), and it is unpleasant but not impossible to find its roots exactly; the only valid root of the original equation is

\[
c = -\frac{1}{3} + \frac{1}{3} \sqrt[3]{\frac{1051 - 15\sqrt{249}}{2}} + \frac{1}{3} \sqrt[3]{\frac{1051 + 15\sqrt{249}}{2}} \approx 5.01556 \ldots.
\]
Consider the rational function

$$f(x) = \frac{x^5 - x^4 - 2x^3}{x^4 - 3x^3 - x^2 + 3x}$$

- For what values of $a$ does $f$ have a removable discontinuity at $a$? What is $\lim_{x \to a} f(x)$ at those $a$?
- For what values of $a$ does $f$ have an infinite discontinuity at $a$?
- What is $\lim_{x \to +\infty} f(x)$?

(Hint: Factor the numerator and the denominator.)

**Solution:** We factor the numerator and denominator of $f$ to obtain

$$f(x) = \frac{x^3(x + 1)(x - 2)}{x(x + 1)(x - 1)(x - 3)}.$$

Away from $x = 0$ and $x = -1$, $f(x)$ simplifies to

$$\frac{x^2(x - 2)}{(x - 1)(x - 3)}.$$  

which is defined and continuous at these $x$-values. Thus, $f$ has removable discontinuities at $x = 0$ and at $x = -1$. From this form of $f$, we also compute that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^2(x - 2)}{(x - 1)(x - 3)} = \frac{0^2(-2)}{(-1)(-3)} = 0 \quad \text{and}$$

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2(x - 2)}{(x - 1)(x - 3)} = \frac{(-1)^2(-3)}{(-2)(-4)} = \frac{3}{8}.$$

At $x = 1$ and at $x = 3$, however, the numerator of the original function is not 0, so $f$ has infinite singularities (and vertical asymptotes) at these $x$-values.

Finally, we observe that

$$\lim_{x \to +\infty} \frac{x^5 - x^4 - 2x^3}{x^4 - 3x^3 - x^2 + 3x} = \lim_{x \to +\infty} \frac{x - 1 - \frac{2}{x}}{1 - \frac{3}{x} - \frac{1}{x^2} + \frac{3}{x^3}} = +\infty.$$
6. Find the value of $a$ such that

$$\lim_{x \to -1} \frac{2x^2 - ax - 14}{x^2 - 2x - 3}$$

exists. What is the value of the limit?

Solution: We observe that $\lim_{x \to -1} x^2 - 2x - 3 = 1 + 2 - 3 = 0$, so in order for this limit to exist, we need the limit of the numerator as $x \to -1$ to be 0 as well. Since

$$\lim_{x \to -1} 2x^2 - ax - 14 = 2 + a - 14 = a - 12,$$

$a - 12 = 0$, and $[a = 12]$. Then, away from $x = -1$,

$$\frac{2x^2 - 12x - 14}{x^2 - 2x - 3} = \frac{2(x - 7)(x + 1)}{(x - 3)(x + 1)} = \frac{2(x - 7)}{(x - 3)}.$$

As $x \to 1$, we see by evaluation that the limit is $\frac{2(-8)}{(-4)} = 4$.