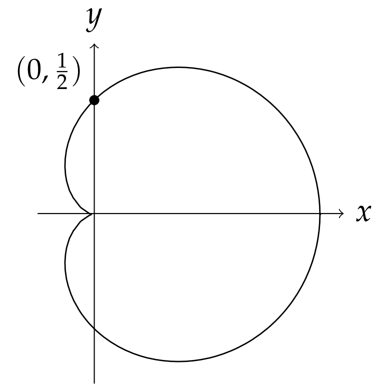
Problems: Extrema, Implicit Differentiation, Related Rates

1. A curve called a *cardioid* is described by the equation

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2.$$

Find the tangent line to the curve at the point $(0, \frac{1}{2})$.



cardioid

Solution: We differentiate the equation implicitly, with respect to x:

$$2x + 2y\frac{dy}{dx} = 2(2x^2 + 2y^2 - x)\left(4x + 4y\frac{dy}{dx} - 1\right)$$

Cancelling the factors of 2 on both sides, and separating the $\frac{dy}{dx}$ terms from the other terms, we have

$$y\frac{dy}{dx} - 4y(2x^2 + 2y^2 - x)\frac{dy}{dx} = (2x^2 + 2y^2 - x)(4x - 1) - x$$

Isolating $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{(2x^2 + 2y^2 - x)(4x - 1) - x}{y - 4y(2x^2 + 2y^2 - x)}.$$

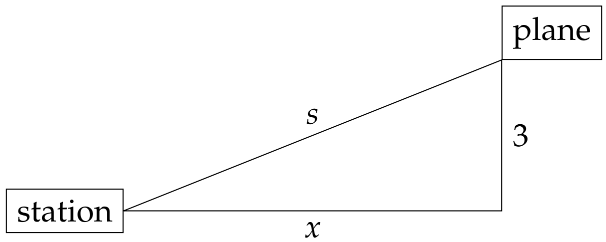
We now evaluate $\frac{dy}{dx}$ at $(0, \frac{1}{2})$ to get the slope of the tangent line. At this point, $2x^2 + 2y^2 - x = 0 + 2(\frac{1}{2})^2 - 0 = \frac{1}{2}$, so

$$\frac{dy}{dx} = \frac{(\frac{1}{2})(-1) - 0}{\frac{1}{2} - 4(\frac{1}{2})(\frac{1}{2})} = \frac{-\frac{1}{2}}{-\frac{1}{2}} = 1.$$

Then the tangent line is $\boxed{y - \frac{1}{2} = 1(x - 0),}$ which simplifies to $\boxed{y = x + \frac{1}{2}.}$

2. A plane flying at an altitude of 3 miles will pass directly over a radar station. The radar station measures that, at time T, the distance between the plane and the station is 5 miles, and the plane is approaching the station at 200 miles per hour. What is the speed of the plane relative to the ground?

Solution: We make a diagram of the distances involved and label the quantities:



Hence, s is the distance from the station to the plane, and x is the distance to the plane along the ground. We wish to find $\frac{dx}{dt}$. From the diagram,

$$s^2 = x^2 + 9,$$

so differentiating with respect to t gives

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}.$$

At this particular time, $s = 5$, and $\frac{ds}{dt} = -200$. We compute that $x = \sqrt{s^2 - 9} = \sqrt{25 - 9} = 4$, so $\frac{dx}{dt} = \frac{5}{4}(-200) = -250$ miles per hour. Hence, the speed of the plane is $\boxed{\text{250 mph.}}$

3. Let $f(x) = \dfrac{3x+4}{x^2+1}$, defined on $(-\infty, \infty)$.

(a) Find the critical numbers of $f(x)$.

(b) Which critical numbers correspond to local maxima? Local minima? Justify your answer using the First or Second Derivative Tests.

(c) What are the absolute maximum and minimum values of f, if they exist? Explain.

Solution:

(a) We compute the derivative of $f(x)$:

$$f'(x) = \frac{(x^2+1)(3) - (3x+4)(2x)}{(x^2+1)^2} = \frac{3 - 8x - 3x^2}{(x^2+1)^2}.$$

This is defined for all real x, so we have no critical numbers from $f'(x)$ not existing. $f'(x) = 0$ exactly when $3 - 8x - 3x^2 = 0$. To find the roots of this quadratic polynomial, we apply the quadratic formula:

$$x = \frac{8 \pm \sqrt{64 - 4(-3)(3)}}{-6} = \frac{-8 \pm \sqrt{100}}{6} = \frac{-4 \pm 5}{3} = -3, \frac{1}{3}.$$

Therefore, $\boxed{x = -3 \text{ and } x = \frac{1}{3}}$ are the critical numbers of f.

(b) We first characterize these critical points via the First Derivative Test. For $x < -3$, $f'(x)$ has the same sign. Since

$$f'(-4) = \frac{3 - 8(-4) - 3(-4)^2}{((-4)^2+1)^2} = \frac{-13}{17^2} < 0,$$

$f'(x)$ is negative for $x < -3$. Similarly, $f'(0) = \dfrac{3}{1^2} = 3 > 0$, so $f'(x)$ is positive between -3 and $\frac{1}{3}$. Finally, $f'(1) = \frac{3-8-3}{2^2} = -2 < 0$, so $f'(x)$ is negative for $x > \frac{1}{3}$.
Since $f'(x)$ changes from $-$ to $+$ at $x = -3$, f has a $\boxed{\text{local minimum}}$ there. Since $f'(x)$ changes from $+$ to $-$ at $x = \frac{1}{3}$, f has a $\boxed{\text{local maximum}}$ there.
Alternately, we could try the Second Derivative Test. Writing $f'(x) = (3 - 8x - 3x^2)(x^2+1)^{-2}$, we compute

$$\begin{aligned} f''(x) &= (-8-6x)(x^2+1)^{-2} + (3 - 8x - 3x^2)(-2)(2x)(x^2+1)^{-3} \\ &= \frac{(-8-6x)(x^2+1) - 4x(3 - 8x - 3x^2)}{(x^2+1)^3} = \frac{2(3x^3 + 12x^2 - 9x - 4)}{(x^2+1)^3}. \end{aligned}$$

Since $(x^2+1)^3 > 0$, the sign of $f''(x)$ is the same as the sign of $g(x) = 3x^3 + 12x^2 - 9x - 4$. Then

$$g(-3) = -81 + 108 + 27 - 4 = 50 > 0 \quad \text{and} \quad g\left(\frac{1}{3}\right) = \frac{1}{9} + \frac{4}{3} - 3 - 4 = -\frac{50}{9} < 0,$$

confirming the statements from the First Derivative Test.

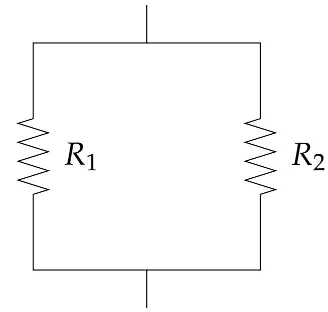
(c) We first compute the values of f at the local extrema:

$$f(-3) = \frac{3(-3)+4}{(-3)^2+1} = \frac{-5}{10} = -\frac{1}{2} \quad \text{and} \quad f\left(\frac{1}{3}\right) = \frac{3(\frac{1}{3})+4}{(\frac{1}{3})^2+1} = \frac{5}{\frac{10}{9}} = \frac{9}{2}.$$

Next, we know that f decreases on the interval $(\frac{1}{3}, +\infty)$, so we check to see how far it decreases. Since f is a rational function where the degree of the denominator is greater than the degree of the numerator, $f(x) \to 0$ as $x \to \infty$. Hence, the values of $f(x)$ on $(\frac{1}{3}, +\infty)$ stay between 0 and $\frac{9}{2}$.
Likewise, f is decreasing on $(-\infty, -3)$, so the values of f on that interval stay between 0 and $-\frac{1}{2}$. Consequently, the absolute minimum of f is $\boxed{-\frac{1}{2},}$ occurring at $\boxed{x = -3,}$ and the absolute maximum is $\boxed{\frac{9}{2},}$ occurring at $\boxed{x = \frac{1}{3}.}$

4. If two resistors with resistances R_1 and R_2 are wired in parallel, the total equivalent resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$



Suppose that at time T, $R_1 = 30\ \Omega$ and is increasing at $3\ \Omega/\text{s}$, and that $R_2 = 60\ \Omega$ and is decreasing at $3\ \Omega/\text{s}$.

(a) How fast does R change with respect to time at time T?

(b) What rate of change of R_2 would make $\frac{dR}{dt} = 0$?

Solution:

(a) We differentiate the equation with respect to t and apply the chain rule:

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt}.$$

We have that $R_1 = 30$, $\frac{dR_1}{dt} = 3$, $R_2 = 60$, and $\frac{dR_2}{dt} = -3$. To solve for $\frac{dR}{dt}$, we also compute $\frac{1}{R} = \frac{1}{30} + \frac{1}{60} = \frac{1}{20}$. Multiplying both sides of the equation by $-(20)^2$ gives

$$\frac{dR}{dt} = \frac{20^2}{30^2}(3) + \frac{20^2}{60^2}(-3) = \frac{4}{9}(3) + \frac{1}{9}(-3) = \frac{4}{3} - \frac{1}{3} = \boxed{1\ \frac{\Omega}{\text{s}}}$$

(b) From above, we have

$$\frac{1}{R^2}\frac{dR}{dt} = \frac{1}{R_1^2}\frac{dR_1}{dt} + \frac{1}{R_2^2}\frac{dR_2}{dt}.$$

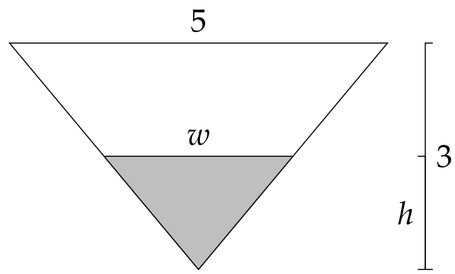
We plug in $R = 20$, $R_1 = 30$, $R_2 = 60$, and $\frac{dR_1}{dt} = 3$, along with the desired rate of change $\frac{dR}{dt} = 0$:

$$\frac{1}{20^2}(0) = \frac{1}{30^2}(3) + \frac{1}{60^2}\frac{dR_2}{dt}$$

Then $\frac{dR_2}{dt} = -\frac{60^2}{30^2}(3) = -(4)(3) = -12$. Hence, R_2 would have to $\boxed{\text{decrease}}$ at a rate of $\boxed{12\ \Omega/\text{s}}$ for R to stay constant.

5. A trough is 12 feet long and has a cross-section shaped like an isosceles triangle, with width 5 feet at the top and height 3 feet. If the trough is filled with water at a rate of 10 ft^3/min, how fast is the water level rising when the water is 2 feet deep?

Solution: We first make a diagram of the cross-section of the trough, labeling the quantities involved:



Here, h and w are the height and width of the water in the trough. From the problem statement, the length of the trough is $l = 12$. Denoting the volume as V, we know $\dfrac{dV}{dt}$ is 10, and we wish to find $\dfrac{dh}{dt}$. Hence, we try to relate h to V.

The water cross-section has area $A = \frac{1}{2}wh$, so the volume of water in the trough is

$$V = Al = \frac{1}{2}wh(12) = 6wh.$$

We try to rewrite w in terms of V and/or h. From the diagram, the water cross-section is similar to the cross-section of the entire trough, so

$$\frac{w}{h} = \frac{5}{3},$$

and $w = \frac{5}{3}h$. Hence,

$$V = 6\left(\frac{5}{3}h\right)h = 10h^2.$$

Differentiating with respect to the time variable t, we have

$$\frac{dV}{dt} = 20h\frac{dh}{dt}.$$

At this point in time, $\dfrac{dV}{dt} = 10$, and $h = 2$, so

$$\frac{dh}{dt} = \frac{1}{20(2)}(10) = \boxed{\frac{1}{4}\ \frac{\text{ft}}{\text{min}}}$$

Therefore, the water level is rising at 3 inches per minute.

6. Let $f(x) = x^{\sqrt{x}}$, defined for $x > 0$.

(a) Compute $f'(x)$.

(b) Find the critical numbers of $f(x)$. Which ones correspond to local minima? Local maxima?

(c) Find the absolute maximum and minimum of $f(x)$ on the interval $\left[\frac{1}{16}, 4\right]$.

Solution:

(a) We compute that $\ln(f(x)) = \ln(x^{\sqrt{x}}) = \sqrt{x}\ln x$. Differentiating,

$$\frac{f'(x)}{f(x)} = \frac{\sqrt{x}}{x} + \frac{1}{2\sqrt{x}}\ln x = \frac{2+\ln x}{2\sqrt{x}} \quad \Rightarrow \quad \boxed{f'(x) = \frac{x^{\sqrt{x}}(2+\ln x)}{2\sqrt{x}}}.$$

(b) $f'(x)$ has the same domain ($x > 0$) as f, so we get no critical numbers from the nonexistence of f'. We then set $f'(x) = 0$. Since $\frac{x^{\sqrt{x}}}{2\sqrt{x}}$ is always positive for $x > 0$, we have that $\ln x + 2 = 0$, so $\ln x = -2$, and $\boxed{x = e^{-2}}$ is the only critical number.

We apply the First Derivative Test to this critical point. At $x = 1$, on the right side of $x = e^{-2}$,

$$f'(1) = \frac{1^1}{2\sqrt{1}}(\ln(1) + 2) = 1(0+2) = 2.$$

On the left side, we note that $e^{-2} \approx 0.135$, so we pick $x = \frac{1}{16} = 0.0625$. Then, noting that $\frac{1}{16} = 2^{-4}$ and that $\ln 2 \approx 0.7$,

$$f'\left(\frac{1}{16}\right) = \frac{(2^{-4})^{\sqrt{2^{-4}}}}{2\sqrt{2^{-4}}}(\ln(2^{-4}) + 2) = \frac{2^{-4(\frac{1}{4})}}{2(\frac{1}{4})}(2 - 4\ln 2) = 2 - 4\ln 2 \approx -0.8.$$

Hence, $f'(x)$ changes from negative to positive at $x = e^{-2}$, so f has a $\boxed{\text{local minimum}}$ there.

Alternately, we can check the sign of $f''(x)$ at $x = e^{-2}$. We use logarithmic differentiation to compute $f''(x)$:

$$\begin{aligned}
\ln(f'(x)) &= \sqrt{x}\ln x + \ln(2+\ln x) - \ln 2 - \frac{1}{2}\ln x \\
\frac{f''(x)}{f'(x)} &= \frac{2+\ln x}{2\sqrt{x}} + \frac{1}{x(2+\ln x)} - \frac{1}{2x} \\
f''(x) &= f'(x)\left(\frac{2+\ln x}{2\sqrt{x}} - \frac{1}{2x}\right) + \frac{x^{\sqrt{x}}}{2x\sqrt{x}}.
\end{aligned}$$

At $x = e^{-2}$, $f'(e^{-2}) = 0$, so the first term vanishes. Then

$$f''(e^{-2}) = 0 + \frac{(e^{-2})^{\sqrt{e^{-2}}}}{2e^{-3}} = \frac{1}{2}e^{3-2/e}.$$

Since $f''(e^{-2})$ is positive, f has a local minimum at $x = e^{-2}$.

(c) We compute the value of f at $x = e^{-2}$ and at the endpoints $x = \frac{1}{16}$ and $x = 4$:

$$\begin{aligned} f(e^{-2}) &= (e^{-2})^{\sqrt{e^{-2}}} = (e^{-2})^{1/e} = e^{-2/e} \approx 0.48, \\ f\left(\frac{1}{16}\right) &= (2^{-4})^{\sqrt{2^{-4}}} = (2^{-4})^{\frac{1}{4}} = 2^{-1} = \frac{1}{2}, \\ f(4) &= 4^{\sqrt{4}} = 4^2 = 16. \end{aligned}$$

Hence, the minimum value is $\boxed{e^{-2/e}, \text{ at } x = e^{-2},}$ and the maximum is $\boxed{16, \text{ at } x = 4.}$